Morphological scale-space preserving transforms in many dimensions

J. Andrew Bangham, Richard Harvey,
Paul D. Ling and Richard V. Aldridge
School of Information Systems
University of East Anglia, Norwich, NR4 7TJ.
Phone: +44 1603 592245, Fax: +44 1603 593345
Email: {ab,rwh,pdl,rva}@sys.uea.ac.uk

Abstract

The theory of an image decomposition that we refer to as a *sieve*, is developed for images defined in any finite number of dimensions. The decomposition has many desirable properties including the preservation of scale-space causality and the localisation of sharp-edged objects in the transformation domain. The decomposition has the additional properties of manipulability, which means that it is easy to construct pattern recognition systems, and scale-calibration which means that it may be used for accurate measurement.

1 Introduction

The use of scale-space for the analysis of images is well established and there is an emerging interest in incorporating scale-space processors as part of high-level computer vision tasks ([1] and [2] for example).

Scale-space vision is usually associated with diffusion based systems [3, 4, 5] in which the image forms the initial conditions for a discretization of the diffusion equation $\nabla^2 I = I_t/D$. In linear scale-space processors the diffusion parameter, D, is a constant and, although the system has the property that it does not enhance regional extrema and so should not introduce edges as scale increases (preserves *scale-space causality*); it blurs the edges of objects in the scene. Since edges are thought to be important for most high-level vision operations, a variant, in which D is made to vary with the magnitude of the gradient of the brightness function, is sometimes used.

A goal of diffusion-based scale-space systems has been to produce results that are "semantically meaningful," for which, in addition to scale-space causality, two further properties have been enunciated [6], *immediate localisation* and *piecewise smoothing*. Immediate localisation is a requirement that the "region boundaries should be sharp and coincide with semantically meaningful boundaries at that resolution" and piecewise smoothing requires that "intra-region smoothing should occur preferentially over inter-region smoothing."

To these properties one might also add that the system should be *scale-calibrated* so that at a particular scale one should see features of only that scale and that it should be possible to *manipulate* the image in scale-space.

Of these properties, causality, scale-calibration and manipulatability are key, since they may be expressed mathematically. This paper presents a system that has these key properties and the others in a number of practical cases. The system has similarities with morphological image processing systems.

Mathematical morphology [7, 8, 9] is based on the analysis of shape and has been developed separately from the processors mentioned above. However, a recent welcome development has been the effort to achieve some unification of scale-space and morphology [10, 11, 12, 13]. Of particular relevance is the granulometry used to characterise size distributions of rock particles [7] and shapes [14, 15]. Granulometry algorithms are fast in one-dimension, [16] and preserve scale-space causality [12] but in higher dimensions do not [13]. To preserve scale-space causality in higher dimensions it is necessary to use connected set granulometries [17, 18, 19, 20, 21].

It is characteristic of morphological filters, such as openings and closings, that intensity maxima and minima are handled by complementary operators, in other words, the filters are not self-dual. Whilst this asymmetry can be an advantage, it results in the filtered representation of an image being susceptible to changes introduced by random noise and clutter. The problem is substantially ameliorated when the asymmetry is reduced by using alternating sequential filters or eliminated by using median-based filters.

Alternating sequential filters with greyscale structuring elements, have been used as pattern matching tools [22]. Whilst suitable for this purpose, in general they are less suitable for decomposing images since, with rigid, flat, structuring elements [23], they are slow to compute and, except when operating in one dimension [24, 25] do not have the key properties of causality and manipulability referred to earlier. Furthermore, in all but one dimension they impose a distortion due to the shape of the structuring element.

It turns out that one-dimensional alternating sequential filters with rigid, flat, structuring elements are a subset of a special class of filters we refer to as *sieves* [26].

Not all alternating sequential filters are sieves and sieves are not a special type of alternating sequential filter – a set of recursive median filters may be formed into a sieve [27], for example.

Alternating sequential filters, composed from reconstruction filters, merge flat zones without creating new ones [20, 28]. There is experimental evidence that one may manipulate the equivalent of the granulomety [26]. We infer that they preserve scale-space causality and, furthermore, that it would extend to area-based alternating sequential filters [17, 18, 19, 20, 29].

Here, we focus on these key properties of causality and manipulatability to draw both aspects of the theory together in any finite number of dimensions.

First we recapitulate some minor results on the operators from which the sieve is constructed, then we show that connected-set sieves preserve scale-space causality and finally demonstrate that a sieve transform, obtained in this way, is also invertible in a sense that is useful for designing segmentation and recognition systems. In two dimensions the connected set sieve decomposes an image in terms of intensity extrema of increasing scale area. There is a clear relationship between this and the requirements of an image analyser laid down in [30]. In three dimensions, the system becomes a volume sieve and we give an example of how it may be used to segment volumes in images of the brain obtained from computed tomography.

2 Definition of M- and N- sieves

An arbitrary array of pixels, or voxels, may be described by a connected graph [31, 32], $G = (V, E)$ where V is the set of vertices and E is the set of pairs that describe the edges.

For example in Figure 1, $V = \{1, 2, \ldots, 16\}$ and $E = \{\{1, 2\}, \{1, 5\}, \ldots, \{15, 16\}\}$. This notation is well established. The necessary definitions, properties of graphs and formal proofs of the following theorems are given in Section 7, but what follows is a summary. From Definition 7.2 when G is a graph and $r \geq 1$, let $\mathcal{C}_r(G)$ denote the set of connected subsets of G with r elements. When $x \in V$, let

$$\mathcal{C}_r(G, x) = \{\, \xi \in \mathcal{C}_r(G) \mid x \in \xi \,\}.$$

For example, in Figure 1, $\mathcal{C}_2(G, 1) = \{\{1, 2\}, \{1, 5\}\}$. All subsequent operations take place over such sets. The structure of an element of $\mathcal{C}_r$ is determined by the adjacency of image pixels. In three dimensions the connected subsets of G would normally be six-connected.

This definition allows a compact definition of an *opening*, ψ_r, and *closing*, γ_r, of size r, which follows that proposed for graphs and connected sets [32, 28, 20, 21]. In Definition 7.15 the operators, $\psi_r, \gamma_r, \mathcal{M}^r, \mathcal{N}^r : \mathbf{Z}^V \to \mathbf{Z}^V$, are defined for each integer, $r \geq 1$, as

$$\begin{aligned}
\psi_r f(x) &= \max_{\xi \in \mathcal{C}_r(G,x)} \min_{u \in \xi} f(u), \\
\gamma_r f(x) &= \min_{\xi \in \mathcal{C}_r(G,x)} \max_{u \in \xi} f(u),
\end{aligned}$$

and

$$\mathcal{M}^r = \gamma_r \psi_r, \qquad \mathcal{N}^r = \psi_r \gamma_r.$$

Thus $\mathcal{M}^r$ is an opening followed by a closing, both of size r and in any finite dimensional space.

The M- and N-sieves of a function, $f \in \mathbf{Z}^V$ are defined in Definition 7.33 as sequences $(f_r)_{r=1}^{\infty}$ with the M- and N-sieves being:

(a) $f_1 = \mathcal{M}^1 f = f$, and $f_{r+1} = \mathcal{M}^{r+1} f_r$;

(b) $f_1 = \mathcal{N}^1 f = f$, and $f_{r+1} = \mathcal{N}^{r+1} f_r$;

both for integers, $r \geq 1$.

These M- and N-sieves are alternating sequential filters [32, 28, 20] but, as described in Section 1 not all alternating sequential filters have the properties of sieves.

The output of M- and N-sieves at any particular stage in the sequence, r, is r-*clean* (Theorem 7.34). The term r-clean is defined in Section 7.2 but means that extremal level connected sets have r or more pixels. The sieve has the effect of locating intensity extrema and "slicing-off peaks and troughs" to produce *flat zones* [20] of r or more pixels hence r-clean, a function from which the flat zones have been "cleaned." Section 7.2 also introduces r-maxclean and r-minclean (see Definitions 7.20). These are the same as the notions of "opened" and "closed" with respect to a particular structuring element [20], but in place of a structuring element, we have a scale parameter r that might refer to area or volume. A function that is r-clean is simultaneously "opened" and "closed" to scale r.

Since all the pixels within each extremal connected set have the same intensity, a simple graph reduction at each stage can lead to a fast algorithm [28]. The one-dimensional equivalent has order complexity p, where p is the number of pixels. In practice we pad the image boundaries with zeros so a two dimensional n by m image has an additional $2(m + n + 2)$ pixels.

A formal analysis of order complexity for the N-dimensional sieve remains to be done, however, it is important to appreciate that the actual speed depends on the number of flat zones (level connected sets) in the image. As result it is appropriate to find the time taken to completely decompose some real images with and without added random noise (to provide extra small scale detail), Figure 2 which suggests the order complexity to be roughly p where p is the largest scale of interest and the image size ranges from 10 by 10 to 500 by 500 pixels.

3 Properties of M- and N-sieves

In one-dimension the flat zones created by a sieve have a length r. This means that the scale parameter can be used for the precise measurement of features [33]. For a two-dimensional image with regular pixelation the flat zones created by the sieve have a defined area and is referred to as an area decomposition c.f. [17]. In three dimensions it is a decomposition by volume. Therefore, area and volume can be measured by these systems. In this, they differ from diffusion based filter systems where length, area and volume estimates have to be calibrated to allow for variations in image intensity.

The causality requirement is that no new extrema or associated edges are generated by any stage of an M- or N-sieve. That is, for for a particular edge, $\{x, y\} \in E$, as r increases, the change in image intensity, $f_r(y) - f_r(x)$, does not change sign and its absolute value never increases. In particular, if it vanishes for some r, it is then zero for all larger r. Theorem 7.36 supposes that $(f_r)_{r=1}^{\infty}$ is the M- or the N-sieve of an $f \in \mathbf{Z}^V$, that $\{x, y\} \in E$ and $\delta_r = f_r(y) - f_r(x)$, for each, r. Then $\delta_1 \geq \delta_2 \geq \cdots \geq 0$, or $\delta_1 \leq \delta_2 \leq \cdots \leq 0$. This result is stronger than that associated with diffusion systems which do not enhance local extrema, but may still introduce extrema.

It has been proposed that signals (such as images) can be re-represented in terms of a set of differences between successive sieve outputs [34, 35]. These differences are called *granules* and it is proposed that objects can be recognised in the granularity domain [36, 37]. Whilst this is similar to granulometry [15, 14] the considerably reduced asymmetry associated with alternating sequential and median sieves, means that one does not have to have “positive” and “negative” scales associated with openings and closings. Furthermore, granularity is less sensitive to noise and clutter than granulometry. Granules are defined in Definition 7.39 as $d_r = f_r - f_{r+1}$ for each integer, r. There is an equivalent definition for the N-granule decomposition.

Clearly it is not difficult to show that the granularity is the image of the original signal and therefore one can invert the transform in the manner outlined in Theorem 7.49. If $(d_r)_{r=1}^{\infty}$ is the M-granule decomposition of a signal with finite support then the original image can be rebuilt from the granularity $f = \sum_{r=1}^{\infty} d_r$. We are interested in more than this.

For the transform to be of practical value, it is desirable to be able to manipulate $(d_r)_{r=1}^{\infty}$ to some useful end. In other words, there ought to be a set of operations that one might perform on the granules and still be able to reconstruct a usefully modified signal that has, as its granularity, the modified granularity of the original signal. In this paper we concentrate on one, Theorem 7.49, that shows that the intensity, but not the polarity, of any granule may be be changed. Denoting the modified granules as $(\hat{d})_{r=1}^{\infty}$ then a modified signal may be formed by summing the granules $\hat{f} = \sum_{r=1}^{\infty} \hat{d}_r$. Now performing a decomposition on this new signal gives $(\hat{d}_r)_{r=1}^{\infty}$ as the M-granule decomposition of $\hat{f}$.

As result, a pattern processor may be produced by: decomposition using M^r or N^r; selectively altering granules (filtering in the granularity domain); and rebuilding, via the above. Such a filter is idempotent and such an approach has already been shown to be successful in one dimension [38, 36].

4 Results

Figure 3(A) shows an image of a doll manually segmented from a larger image.

It is shown as a topological relief in Figure 3(B). The spiky detail, particularly evident around the head, represents small scale regional extremes of area that are removed by sieving to scale 4, as in Figure 3(B)ii. The small extrema that are removed are shown in Figure 3(C)i (actually $I - M^4(I)$). Likewise the pale highlight on the (left side) of the hair, is a feature represented by a larger scale extremum that is removed with a larger, scale 25, area sieve. The differences between Figure 3(B)ii and (B)iii are shown in Figure 3(C)ii. No new features can be seen in Figure 3(B)iii and what remains is a set of large flat zones with edges in the same places as those in the original. This is consistent with Theorem 7.36 and Theorem 7.34. This localisation of features is an important advantage over, say, Gaussian scale-space which blurs the edges.

The process is essentially regional and is not equivalent to redefining the intensity map over the whole image. Nor does it alter the corners of objects in the way that a Gaussan, morphological or median filter with a rigid structuring element, or window, would be expected to.

However, there is a problem. In Figure 3(A) there are over 5,000 pixels and so, potentially, 5,000 different area scales. In practice, this is often far too many. An alternative is shown in Figure 4 (bottom right panel). At the bottom is an image of four pieces of string. The two longer pieces are the same length. The image is decomposed into 17 area *channels*. Each channel is formed as the partial sum of granules from a range of scales, $c_i = \sum_{r=r_i}^{r=r_{i+1}} d_r$,where r_i increases logarithmically, $r_{i+1} = 2r_i - 1$. Most of the activity due to the longer pieces of string is observed in channel 10, whilst that for the two smaller pieces can be seen in channels 7 and 8. Of course, since the pieces of string are not of uniform intensity, and there are shadows, activity is not confined to a single channel, rather it peaks at the appropriate scale.

This result would be difficult to achieve using a filter with a structuring element (for example, an alternating sequential filter by reconstruction [39] with square structuring element) because, a structuring element chosen to encompass the small bits of string would remove the more coiled of the long strings. The area sieve does not rely on a match between the shape of a structuring element and an object. The strings can be arranged in any shape (although, if the string forms loops it might be necessary to use a sieve that takes advantage of the sign of the local extremum, cf. [17]).

Figure 5 shows a more complicated example: a human face. Note the nostrils in channel 5, the eyes in channels 6 and 7, the mouth in channels 9 and 10 and the entire face in channel 15. The larger scale channels can be used for segmentation, for example, channel 15 could be used as a mask to pick out facial features from the background for, by Theorem 7.36, it accurately represents features in the original image.

A simple example of how such an area decomposition might be used is shown in Figure 6. It shows, in column (a), four frames taken from a sequence of a speaker saying the letter “A” together with area channels 9, 10 and 11. As her mouth opens, so the activity associated with the mouth moves from a peak in channel 9 to a peak in channel 10/11 and back. In other words, it is possible to track the mouth opening through changes in channel activity. In practice, we find that a combination of one and two dimensional granularities yields feature vectors that help visual speechreading considerably [40].

The sieve can also be of value in three dimensions. Figure 7 shows a set of transverse scans of a human head generated by X-ray tomography. The sections start at the level of the nose and go up to the top of the head. A guide frame, which is opaque to X-rays, is attached to the head. The white, calcified bony tissue of the skull is readily identified and could be segmented out by simple thresholding, however, we have chosen to use a very large scale volume sieve.

Figure 8(A) shows a rendered reconstruction of the volume channel containing between 25,600 and 51,200 voxels. The left cheekbone can be seen looping out at the bottom of the image, and the upper part of the nose is visible at the bottom left. The guide frame fastenings protrude from the skull.

The third ventricle is a complex shaped void within the brain. It appears as a darkened region circled in panel 16 of Figure 7 and runs up through the brain, bifurcating to form two “wings” that come together at the top. To visualise this three dimensional structure the primary segmentation is made by taking a volume channel encompassing 800 to 1600 voxels. This first step achieves a near perfect segmentation but some other features, of the same volume, are also present. Since they do not penetrate to the same depth, they occupy a larger area of those slices in which they are present and so they can be removed by area sieving each slice to remove areas larger than 100 pixels. This leaves some small regions, which are no longer connected through the slices, that can be removed by re-sieving through the original volume sieve. As might be expected from Theorem 7.49 removing some granules, but leaving others alone, allows the third ventricle to be seen unchanged.[1].

5 Conclusions

A nonlinear method of generating scale-space has been presented. It has the following features:

1. it preserves scale-space causality;
2. sharp-edged objects in the original image remain precisely localised;
3. it operates on images defined in any finite dimension;
4. it is scale-calibrated;
5. the image may be processed in the scale-space domain;
6. the decomposition appears to offer semantically meaningful feature vectors.

The empirical evidence suggests that the area sieve robustly rejects random noise [41]. Although the theory has been presented for M- and N-sieves our experimental evidence suggests that one can form a higher-dimensional sieves using an analogue of the one-dimensional recursive median sieve and they have the above properties. Initial studies show that the n-dimensional sieve can form a valuable part of computer vision systems.

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[1] The objects in Figure 8(B) *are* connected but by thin filaments that we have had difficulty rendering

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7 Appendix

7.1 Essential graph theory

The final aim of the theory presented here is the "invertibility" Theorem 7.49. On the way we prove some well known, together with some more recent, results to which we draw attention In this section we shall give some results concerning graphs that are needed to define the M- and N-sieves and M- and N-granule decompositions and prove the various properties thereof.

Graphs have been discussed in connection to mathematical morphology elsewhere [32, 31]. By a graph we shall mean a simple undirected graph. That is, we make the following definition.

Definition 7.1 *A graph is a pair, $G = (V, E)$, where V is a set and E is a set of subsets of V all of which have precisely two elements. The elements of V are known as the vertices of G and the elements of E are known as the edges of G. An edge, $\{x, y\} \in E$, is said to incident with the vertices, x and y, and it is said to joint x and y.*

Definition 7.2 *When $G = (V, E)$ is a graph, a path of G is a finite sequence, $\pi = (x_0, x_1, \ldots, x_l)$, of vertices of G such that $\{x_{i-1}, x_i\} \in E$, for $i = 1, \ldots,$*

l. The length of π is defined to be l and π is also said be a path of G from x_0 to x_l. We let

$$[\pi] = \{x_0, x_1, \ldots, x_l\} \subseteq V.$$

When $A \subseteq V$, the path, π, of G is said to be in A if and only if $[\pi] \subseteq A$.

Definition 7.3 *When $G = (V, E)$ is a graph, $C \subseteq V$ is said to be connected (in G) if and only if, for every $x, y \in C$, there exists a path of G in C (possibly with zero length) from x to y. The graph, G, is said to be connected if and only if V is connected in G.*

It can be seen that when $C, D \subseteq V$ are connected and $C \cap D \neq \emptyset$, we have that $C \cup D$ is connected.

Definition 7.4 *When $G = (V, E)$ is a graph and $A \subseteq V$, a connected component of A is a nonempty maximal connected subset of A. By the connected components of G we mean the connected components of V.*

The connected components of A partition A since they are the equivalence classes of the equivalence relation, $\sim$, on A given by $x \sim y \iff$ there exists a path of G in A from x to y.

Definition 7.5 *When $G = (V, E)$ is a graph and $A \subseteq V$, the neighbours of A are the elements of the set,* $\operatorname{nbr} A$, *which we define by* $\operatorname{nbr} A = \{u \in V \setminus A \mid \{x, u\} \in E$ *for some* $x \in A\}$.

For $x \in V$, we let $\operatorname{nbr} x$ *denote* $\operatorname{nbr}\{x\}$ *and call the elements of* $\operatorname{nbr} x$ *the neighbours of x.*

It is clear that, in the situation of Definition 7.5,

$$\operatorname{nbr} x = \{\, u \in V \mid \{x, u\} \in E \,\},$$

and

$$\operatorname{nbr} A = \Big(\bigcup_{x \in A} \operatorname{nbr} x \Big) \setminus A.$$

Theorem 7.6 *If $G = (V, E)$ is a graph, $A \subseteq V$ and C is a connected component of A, then*

$$A \cap \operatorname{nbr} C = \emptyset.$$

Proof: If $x \in A \cap \operatorname{nbr} C$, then $C \cup \{x\}$ would be a connected subset of A which would contradict that C is a maximal connected subset of A. So $A \cap \operatorname{nbr} C = \emptyset$, as required. □

Theorem 7.7 *If $G = (V, E)$ is a graph and $A, C \subseteq V$ with C connected, $C \cap A \neq \emptyset$ and $C \not\subseteq A$, then there exist $x \in C \cap A$ and $y \in C \setminus A$ such that $\{x, y\} \in E$. So*

$$C \cap \operatorname{nbr} A \neq \emptyset.$$

Proof: Let $u \in C \cap A$ and $v \in C \setminus A$. Then, since C is connected, there exits a path, $\pi = (x_0, x_1, \ldots, x_l)$, in C from u to v. $x_l = v \notin A$. Let k be the least such that $x_k \notin A$. Then $x_{k-1} \in A$ ($k \neq 0$ since $x_0 = u \in A$). Thus we may take $x = x_{k-1}$ and $y = x_k$. We clearly have $y \in C \cap \operatorname{nbr} A$. So $C \cap \operatorname{nbr} A \neq \emptyset$. □

By taking $C = V$ in Theorem 7.7, we immediately obtain the following result.

Theorem 7.8 *If $G = (V, E)$ is a connected graph and $A \subseteq V$ with $A \neq \emptyset, V$, then $\operatorname{nbr} A \neq \emptyset$.*

At this stage we are in a position to introduce Definition 7.9.

Definition 7.9 *When G is a graph and $r \geq 1$, we let $\mathcal{C}_r(G)$ denote the set of connected subsets of G with r elements. When $x \in V$, we let*

$$\mathcal{C}_r(G, x) = \{\, \xi \in \mathcal{C}_r(G) \mid x \in \xi \,\}.$$

Theorem 7.10 *If $G = (V, E)$ is a graph and $C \subseteq V$ is connected, then, for each $x \in C$ and each integer, r, with $1 \leq r \leq |C|$, there exists a $\xi \in \mathcal{C}_r(G, x)$ with $\xi \subseteq C$.*

Note that when C is infinite, we take $|C| = \infty$, which is greater than all integers. So Theorem 7.10 applies in this case for all $r \geq 1$.

Proof of Theorem 7.10: We use induction on r. When $r = 1$ we just take $\xi = \{x\}$. When $r > 1$, there exists, by induction hypothesis, an $\eta \in \mathcal{C}_{r-1}(G, x)$ such that $\eta \subseteq C$. Now $C \not\subseteq \eta$ since $|C| > |\eta|$. So Theorem 7.7 gives that $C \cap \operatorname{nbr} \eta \neq \emptyset$. Let $u \in C \cap \operatorname{nbr} \eta$ and put $\xi = \eta \cup \{u\}$. Then $\xi \in \mathcal{C}_r(G, x)$ and $\xi \subseteq C$, as required. $\square$

Definition 7.11 *When $G = (V, E)$ is a graph, the degree of a node, $x \in V$, is the number of edges incident with x. We say that G has finite degrees if and only if the degree of every node of G is finite.*

Theorem 7.12 *If $G = (V, E)$ is a graph with finite degrees, then* $\operatorname{nbr} A$ *is a finite subset of V for each finite subset, A, of V.*

Proof: $\operatorname{nbr} A = (\bigcup_{x \in A} \operatorname{nbr} x) \setminus A$. But, for each $x \in A$, $\operatorname{nbr} x$ is finite since x has finite degree. Thus $\operatorname{nbr} A$ is finite as required. $\square$

Theorem 7.13 *If $G = (V, E)$ is a graph with finite degrees, then $\mathcal{C}_r(G, x)$ is a finite set, for each $x \in V$ and each $r \geq 1$.*

Proof: We use induction on r. When $r = 1$, $\mathcal{C}_r(G, x)$ has just one element namely $\{x\}$.

Suppose that $r > 1$. Then, for $\xi \in \mathcal{C}_r(G, x)$, we have, by Theorem 7.10, that there exists an $\eta \in \mathcal{C}_{r-1}(G, x)$ such that $\eta \subseteq \xi$. Let $\{u\} = \xi \setminus \eta$. Then $\xi = \eta \cup \{u\}$ and $u \in \operatorname{nbr} \eta$.

So we have shown that each $\xi \in \mathcal{C}_r(G, x)$ is of the form $\eta \cup \{u\}$, where $\eta \in \mathcal{C}_{r-1}(G, x)$ and $u \in \operatorname{nbr} \eta$. By the induction hypothesis, there are only finitely many choices for η and, by Theorem 7.12, for each such η, there are only finitely many choices for u. So $\mathcal{C}_r(G, x)$ must be finite, as required. $\square$

Theorem 7.14 *If $G = (V, E)$ is a connected graph with finite degrees and A is a finite subset of V, then $V \setminus A$ has finitely many connected components.*

Proof: This is clear when $A = \emptyset$ since $V \setminus A = V$ is connected. So suppose that $A \neq \emptyset$.

Suppose that C is a connected component of $V \setminus A$. Then, by Theorem 7.6, $(V \setminus A) \cap \operatorname{nbr} C = \emptyset$. But, since $C \neq \emptyset, V$, Theorem 7.8 gives that $\operatorname{nbr} C \neq \emptyset$. So $A \cap \operatorname{nbr} C \neq \emptyset$.

Let $x \in A \cap \operatorname{nbr} C$ and take $y \in C$ so that $\{x, y\} \in E$. Then, since $y \in C \subseteq V \setminus A$, we have that $y \in \operatorname{nbr} A$. So we have shown that every connected component of $V \setminus A$ contains an element, y, of $\operatorname{nbr} A$.

By Theorem 7.12, $\operatorname{nbr} A$ is finite. So, since the connected components of $V \setminus A$ are pairwise disjoint, there can be only finitely many connected components of $V \setminus A$, as required. □

7.2 The Operators and their Properties

From now on in this paper, we shall assume that $G = (V, E)$ is a connected graph with finite degrees. We shall also assume that V is infinite. The operators $\psi_r, \gamma_r, \mathcal{M}^r, \mathcal{N}^r \colon \mathbf{Z}^V \to \mathbf{Z}^V$ have been discussed in the context of area based granulometry and connected sets [42, 17, 18, 19]. Here, we introduce them in Definition 7.15 Definition 7.9 allows a compact definition of an *opening*, ψ_r, and *closing*, γ_r, of size r.

Definition 7.15 (The main operators) *For each integer, $r \geq 1$, the operators, $\psi_r, \gamma_r, \mathcal{M}^r, \mathcal{N}^r \colon \mathbf{Z}^V \to \mathbf{Z}^V$ are defined by*

$$\begin{aligned}
\psi_r f(x) &= \max_{\xi \in \mathcal{C}_r(G,x)} \min_{u \in \xi} f(u), \\
\gamma_r f(x) &= \min_{\xi \in \mathcal{C}_r(G,x)} \max_{u \in \xi} f(u),
\end{aligned}$$

and

$$\mathcal{M}^r = \gamma_r \psi_r, \qquad \mathcal{N}^r = \psi_r \gamma_r.$$

(ψ_r and γ_r are well defined since, for each $x \in V$, $\mathcal{C}_r(G, x)$ is nonempty by Theorem 7.10 and finite by Theorem 7.13.)

Thus $\mathcal{M}^r$ is an opening followed by a closing, both of size r and in any finite dimensional space. From Definition 7.13 comes the following result ((e) follows from (d)).

Theorem 7.16 *Suppose that $f, g \in \mathbf{Z}^V$. Then, for each integer, $r \geq 1$, we have the following.*

(a) $\psi_1 = \gamma_1 = \mathcal{M}^1 = \mathcal{N}^1 = \iota$, *the identity operator.*

(b) $\gamma_r f = -\psi_r(-f)$ *and* $\mathcal{N}^r f = -\mathcal{M}^r(-f)$.

(c) $\psi_r f \leq f \leq \gamma_r f$.

(d) $f \leq g \Rightarrow \psi_r f \leq \psi_r g$ *and* $\gamma_r f \leq \gamma_r g$.

(e) $f \leq g \Rightarrow \mathcal{M}^r f \leq \mathcal{M}^r g$ *and* $\mathcal{N}^r f \leq \mathcal{N}^r g$.

From Theorem 7.16(b), we see that results concerning ψ_r give corresponding results concerning γ_r etc.

Definition 7.17 *When $f \in \mathbf{Z}^V$, a cell of f is a $C \subseteq V$ that is a connected component of $f^{-1}\{\alpha\}$, for some $\alpha \in \mathbf{Z}$.*

We note that, for each $f \in \mathbf{Z}^V$, the cells of f partition V since the family, $(f^{-1}\{\alpha\})_{\alpha \in \mathbf{Z}}$, partitions V.

Theorem 7.18 *If C is a cell of an $f \in \mathbf{Z}^V$, then*[2]

$$f(x) \neq f(C), \qquad \textit{for all } x \in \operatorname{nbr} C. \tag{1}$$

Proof: Let $\alpha = f(C)$. Then C is a connected component of $f^{-1}\{\alpha\}$. So, by Theorem 7.6, $f^{-1}\{\alpha\} \cap \operatorname{nbr} C = \emptyset$. This gives (1). □

In the following we develop the ideas of r-minclean, r-maxclean and r-clean. These are a very slight variant on those of "opened", "closed" and both[8, 9].

Definition 7.19 *When $f \in \mathbf{Z}^V$, a maximum (resp. minimum) cell of f is a cell, C, of f such that $f(x) < f(C)$ (resp. $f(x) > f(C)$), for all $x \in \operatorname{nbr} C$. The maximum and minimum cells of f are known collectively as the extremal cells of f.*

Definition 7.20 *When $f \in \mathbf{Z}^V$, we say, for an integer, $r \geq 1$, that f is r-maxclean (resp. r-minclean) if and only if all maximum (resp. minimum) cells*

[2] When an $f \in \mathbf{Z}^V$ is constant on a nonempty subset, A, of V, we shall write $f(A)$ for the common value, α, on A. Since the image of A under f is $\{\alpha\}$, this is only a slight abuse of notation.

of f have at least r elements. We say that f is r-clean if and only if f is both r-maxclean and r-minclean; i.e. if and only if all extremal cells of f have at least r elements.

Definition 7.21 *For $f \in \mathbf{Z}^V$, we let $\operatorname{inc} f = \{ (x,y) \in V^2 \mid \{x,y\} \in E, f(x) < f(y)\}$, $\operatorname{dec} f = \{ (x,y) \in V^2 \mid \{x,y\} \in E, f(x) > f(y)\}$,*

For $A \subseteq V$, we let $\operatorname{ent} A = \{(x,y) \in V^2 \mid \{x,y\} \in E, x \notin A,\ y \in A\}$, $\operatorname{lve} A = \{(x,y) \in V^2 \mid \{x,y\} \in E, x \in A,\ y \notin A\}$.

The $(x,y) \in V^2$ with $\{x,y\} \in E$ are known as the arcs of G. The elements of $\operatorname{inc} f$ (resp. $\operatorname{dec} f$) are known as the increasing (resp. decreasing) arcs of f. The elements of $\operatorname{ent} A$ (resp. $\operatorname{lve} A$) are known as the arcs entering (resp. leaving) A.

For $Q \subseteq V^2$, we define the transpose, Q^{T}, of Q by $Q^{\mathrm{T}} = \{ (y,x) \mid (x,y) \in Q \}$.

It is clear that for $f \in \mathbf{Z}^V$, $\operatorname{inc} f = (\operatorname{dec} f)^{\mathrm{T}} = \operatorname{dec}(-f)$, $\operatorname{dec} f = (\operatorname{inc} f)^{\mathrm{T}} = \operatorname{inc}(-f)$, and for $A \subseteq V$, $\operatorname{ent} A = (\operatorname{lve} A)^{\mathrm{T}} = \operatorname{lve}(V \setminus A)$, $\operatorname{lve} A = (\operatorname{ent} A)^{\mathrm{T}} = \operatorname{ent}(V \setminus A)$.

We also have the have the following result

Theorem 7.22 *For $f_1, \ldots, f_n \in \mathbf{Z}^V$,*

$$\begin{aligned} &\operatorname{inc}(f_1 + \cdots + f_n) \subseteq \\ &\qquad \operatorname{inc} f_1 \cup \cdots \cup \operatorname{inc} f_n, \end{aligned} \tag{2}$$

$$\begin{aligned} &\operatorname{dec}(f_1 + \cdots + f_n) \subseteq \\ &\qquad \operatorname{dec} f_1 \cup \cdots \cup \operatorname{dec} f_n. \end{aligned} \tag{3}$$

Proof: Suppose that (x,y) is an arc of G that does not belong to the right hand side of (2). Then $f_i(x) \geq f_i(y)$, for $i = 1, \ldots, n$. So

$$f_1(x) + \cdots + f_n(x) \geq f_1(y) + \cdots + f_n(y).$$

Thus (x,y) does not belong to the left hand side of (2). This proves (2). Taking transposes gives (3). □

We have the following characterizations of r-maxclean and r-minclean.

Theorem 7.23 *If $f \in \mathbf{Z}^V$, then, for each integer, $r \geq 1$, we have that* *(a) f is r-maxclean if and only if*

$$\emptyset \neq A \subseteq V, \ \text{ent}\, A \subseteq \text{inc}\, f \ \Rightarrow \ |A| \geq r, \tag{4}$$

(b) f is r-minclean if and only if

$$\emptyset \neq A \subseteq V, \ \text{ent}\, A \subseteq \text{dec}\, f \ \Rightarrow \ |A| \geq r. \tag{5}$$

In order to prove this we need the following lemma.

Lemma 7.24 *If $A \subseteq V$ is nonempty and finite, $f \in \mathbf{Z}^V$ and* $\text{ent}\, A \subseteq \text{inc}\, f$*, then there is a maximum cell, C, of f such that $C \subseteq A$.*

Proof: Let $x \in A$ be such that $f(x)$ is greatest and let C be the cell of f containing x.

If $C \not\subseteq A$, then, by Theorem 7.7, there exit $u \in C \setminus A$ and $v \in C \cap A$ such that $\{u, v\} \in E$. We have that $f(u) = f(v)$ since $u, v \in C$ and C is a cell of f. But we have that $f(u) < f(v)$ since $(u, v) \in \text{ent}\, A \subseteq \text{inc}\, f$. So we must have that $C \subseteq A$.

Suppose that $y \in \text{nbr}\, C$. If $y \in A$, then $f(y) < f(C)$ by Theorem 7.18 and the choice of x. If $y \notin A$, then, since $y \in \text{nbr}\, C$, there exists $z \in C \subseteq A$ such that $\{y, z\} \in E$. Then $(y, z) \in \text{ent}\, A \subseteq \text{inc}\, f$. So $f(y) < f(z) = f(C)$. Thus we have shown that $f(y) < f(C)$, for all $y \in \text{nbr}\, C$; i.e. that C is a maximum cell of f. This completes the proof. $\square$

Proof of Theorem 7.23: Since (b) follows from (a) applied to $-f$, we shall only consider (a).

'if' part: If C is a maximum cell of f, then $\emptyset \neq C \subseteq V$ and $\text{ent}\, C \subseteq \text{inc}\, f$. So, by (4), $|C| \geq r$. Thus f is r-maxclean, as required.

'only if' part: Suppose that the left hand side of (4) holds. We have to show that $|A| \geq r$. This is clear when A is infinite. So assume that A is finite. Then, by Lemma 7.24, there exist a maximum cell, C, of f with $C \subseteq A$. But f is r-maxclean. So $|C| \geq r$ and so, since $C \subseteq A$, $|A| \geq r$, as required. $\square$

Theorem 7.25 *Suppose that $f \in \mathbf{Z}^V$. Then, for each integer, $r \geq 1$, we have the following.*

(a) If f is r-maxclean, then $\psi_r f = f$.

(b) If f is r-minclean, then $\gamma_r f = f$.

(c) If f is r-clean, then $\mathcal{M}^r f = \mathcal{N}^r f = f$.

Proof: (b) will follow from (a) applied to $-f$ and (c) will follow from (a) and (b), since $\mathcal{M}^r = \gamma_r \psi_r$ and $\mathcal{N}^r = \psi_r \gamma_r$. So we need only consider (a).

Suppose that $x \in V$. Let

$$A = \{\, u \in V \mid f(u) \geq f(x) \,\},$$

and let C be the connected component of A containing x. From Theorem 7.6, $A \cap \operatorname{nbr} C = \emptyset$. So

$$f(u) < f(x), \qquad \text{for all } u \in \operatorname{nbr} C. \tag{6}$$

If $(u, v) \in \operatorname{ent} C$, then $f(u) < f(x)$ by (6) and $f(x) \leq f(v)$ since $v \in C \subseteq A$. So $f(u) < f(v)$. Thus $\operatorname{ent} C \subseteq \operatorname{inc} f$.

So, since C is nonempty, Theorem 7.23(a) gives that $|C| \geq r$. So, by Theorem 7.10, there exists a $\xi \in \mathcal{C}_r(G, x)$ with $\xi \subseteq C \subseteq A$. Thus, by Definition 7.15,

$$\psi_r f(x) \geq \min_{u \in \xi} f(u) \geq f(x).$$

So with Theorem 7.16(c), we obtain that $\psi_r f(x) = f(x)$. This proves (a). □

Theorem 7.26 *If $f \in \mathbf{Z}^V$, then, for each integer, $r \geq 1$, $\psi_r f$ is r-maxclean and $\gamma_r f$ is r-minclean.*

Proof: By considering $-f$ in place of f, we see that we need only consider $\psi_r f$.

Suppose, for a contradiction, that $\psi_r f$ is not r-maxclean. Then let C be a maximum cell of $\psi_r f$ with $|C| < r$. Let $x \in C$ and, by Definition 7.15, let $\xi \in \mathcal{C}_r(G, x)$ be such that

$$\min_{u \in \xi} f(u) = \psi_r f(x). \tag{7}$$

Then $\xi \not\subseteq C$ since $|\xi| > |C|$. So, by Theorem 7.7, $\xi \cap \operatorname{nbr} C \neq \emptyset$. Let $y \in \xi \cap \operatorname{nbr} C$. Then $\xi \in \mathcal{C}_r(G, y)$. So, by Definition 7.15, $\psi_r f(y) \geq \min_{u \in \xi} f(u)$. So, by (7), $\psi_r f(y) \geq \psi_r f(x)$. But this contradicts that C is a maximum cell and $y \in \operatorname{nbr} C$. So $\psi_r f$ is r-maxclean, as required. □

From Theorem 7.26, the elements of the range of ψ_r are r-maxclean and so, by Theorem 7.25, unchanged by applying ψ_r. Similar remarks hold for γ_r. So we have the following result.

Theorem 7.27 (Idempotence of open/close) *For each integer, $r \geq 1$, the operators, ψ_r and γ_r are idempotent; i.e.*

$$\psi_r \psi_r = \psi_r, \qquad \textit{and} \qquad \gamma_r \gamma_r = \gamma_r.$$

Theorem 7.28 *If $f \in \mathbf{Z}^V$ and g is one of*

$$\begin{array}{llll} \psi_r f, & \gamma_r f, & \mathcal{M}^r f, & \mathcal{N}^r f, \\ f - \psi_r f, & f - \gamma_r f, & f - \mathcal{M}^r f, & f - \mathcal{N}^r f, \end{array}$$

for some integer, $r \geq 1$, then we have the following.

(a) $\{x, y\} \in E$, $f(x) \leq f(y) \;\Rightarrow\; g(x) \leq g(y)$.

(b) $\{x, y\} \in E$, $f(x) = f(y) \;\Rightarrow\; g(x) = g(y)$.

(c) For each connected $C \subseteq V$, f is constant on C $\Rightarrow$ g is constant on C. So, in particular, g is constant on each cell of f.

(d) $\operatorname{inc} f \supseteq \operatorname{inc} g$.

(e) $\operatorname{inc} f = \operatorname{inc} g \cup \operatorname{inc}(f - g)$.

(f) If f is s-maxclean (resp. s-minclean, s-clean), for some integer, $s \geq 1$, then g is also s-maxclean (resp. s-minclean, s-clean).

In order to prove this we use the following two lemmas.

Lemma 7.29 *If $f \in \mathbf{Z}^V$, then, for each integer, $r \geq 1$, $\psi_r f(y) \geq \min\{\psi_r f(x), f(y)\}$, for all $\{x, y\} \in E$.*

Proof: Suppose that $\{x, y\} \in E$. Then, by Definition 7.15, there exists a $\xi \in \mathcal{C}_r(G, x)$ such that

$$\min_{u \in \xi} f(u) = \psi_r f(x). \tag{8}$$

Now $\xi \cup \{y\}$ is connected and $|\xi \cup \{y\}| \geq r$. So, by Theorem 7.10, there exists an $\eta \in \mathcal{C}_r(G, y)$ such that $\eta \subseteq \xi \cup \{y\}$. Thus, by Definition 7.15 and (8),

$$\begin{aligned} \psi_r f(y) \;&\geq\; \min_{u \in \eta} f(u) \geq \min\{\min_{u \in \xi} f(u), f(y)\} \\ &= \min\{\psi_r f(x), f(y)\}, \end{aligned}$$

as required. $\square$

Lemma 7.30 *If $f, g \in \mathbf{Z}^V$ are such that*

$$\{x,y\} \in E,\ f(x) \leq f(y) \ \Rightarrow\ g(x) \leq g(y), \tag{9}$$

then we have the following.

(a) $\{x,y\} \in E,\ f(x) = f(y) \ \Rightarrow\ g(x) = g(y)$.

(b) For each connected $C \subseteq V$, f is constant on C $\Rightarrow$ g is constant on C. So, in particular, g is constant on each cell of f.

(c) $\operatorname{inc} f \supseteq \operatorname{inc} g$.

(d) If f is s-maxclean (resp. s-minclean, s-clean), for some integer, $s \geq 1$, then g is also s-maxclean (resp. s-minclean, s-clean).

Proof: (a): This follows from (9) and (9) with the roles of x and y interchanged.

(b): Suppose that x and y are any elements of C. Then, since C is connected, there exists a path $\pi = (x_0, x_1, \ldots, x_l)$ in C from x to y. For $i = 1, \ldots, l$, we have that $\{x_{i-1}, x_i\} \in E$ and $f(x_{i-1}) = f(x_i)$. So (a) gives that $g(x_{i-1}) = g(x_i)$. So $g(x) = g(x_0) = \cdots = g(x_l) = g(y)$. So we have shown that $g(x) = g(y)$, for all $x, y \in C$; i.e. g is constant on C, as required.

(c): If $(x,y) \in \operatorname{inc} g$, then we cannot have $f(x) \geq f(y)$ since (9) would then give that $g(x) \geq g(y)$. So $f(x) < f(y)$; i.e. $(x,y) \in \operatorname{inc} f$. So we have shown that $\operatorname{inc} f \supseteq \operatorname{inc} g$.

(d): If f is s-maxclean, then, by Theorem 7.23(a),

$$\emptyset \neq A \subseteq V,\ \operatorname{ent} A \subseteq \operatorname{inc} f \ \Rightarrow\ |A| \geq r.$$

But, by (c), $\operatorname{inc} f \supseteq \operatorname{inc} g$. So

$$\emptyset \neq A \subseteq V,\ \operatorname{ent} A \subseteq \operatorname{inc} g \ \Rightarrow\ |A| \geq r.$$

So, by Theorem 7.23(a), g is s-maxclean.

Similarly, the s-minclean case follows using that $\operatorname{dec} f \supseteq \operatorname{dec} g$ by (c) and by using Theorem 7.23(b). The s-clean case follows by combining the s-maxclean and s-minclean cases. □

Proof of Theorem 7.28: We first consider (a). So suppose that $\{x,y\} \in E$ and $f(x) \leq f(y)$. Then, from Lemma 7.29, $\psi_r f(y) \geq \min\{\psi_r f(x), f(y)\}$. But

$f(y) \geq f(x) \geq \psi_r f(x)$. So we obtain that $\psi_r f(y) \geq \psi_r f(x)$, which proves (a) in the case when $g = \psi_r f$.

The case when $g = \gamma_r f$ follows by applying the ψ_r case to $-f$ with the roles of x and y interchanged.

We have $\psi_r f(x) \leq \psi_r f(y)$ by the ψ_r case. So, by the γ_r case with $\psi_r f$ as f, $\gamma_r \psi_r f(x) \leq \gamma_r \psi_r f(y)$, which proves (a) when $g = \mathcal{M}^r f = \gamma_r \psi_r f$. The $\mathcal{N}^r$ case is similar.

Now from Lemma 7.29, $\psi_r f(x) \geq \min\{f(x), \psi_r f(y)\}$. So $f(x) - \psi_r f(x) \leq f(x) - \min\{f(x), \psi_r f(y)\} = \max\{0, f(x) - \psi_r f(y)\} \leq f(y) - \psi_r f(y)$, which proves (a) in the case when $g = f - \psi_r f$.

The case when $g = f - \gamma_r f$ follows by applying the case when $g = f - \psi_r f$ with $-f$ as f and the roles of x and y interchanged.

Now $f - \mathcal{M}^r f = (f - \psi_r f) + (h - \gamma_r h)$, where $h = \psi_r f$. But $h(x) \leq h(y)$ by the ψ_r case. So we obtain the case when $g = f - \mathcal{M}^r f$ from the cases when $g = f - \psi_r f$ and when $g = f - \gamma_r f$. The case when $g = f - \mathcal{N}^r f$ is similar.

(b)–(d) and (f) now follow from Lemma 7.30. For (e), we have that $\operatorname{inc} f \supseteq \operatorname{inc} g$ by (d) and $\operatorname{inc} f \supseteq \operatorname{inc}(f - g)$ by (d) with $f - g$ as g. So

$$\operatorname{inc} f \supseteq \operatorname{inc} g \cup \operatorname{inc}(f - g).$$

Since $f = g + (f - g)$, Theorem 7.22 gives the reverse inclusion. So (e) holds. □

Theorem 7.31 *If $f \in \mathbf{Z}^V$, then, for each integer, $r \geq 1$, $\mathcal{M}^r f$ and $\mathcal{N}^r f$ are both r-clean.*

Proof: Since $\mathcal{M}^r f = \gamma_r \psi_r f$, we have that $\mathcal{M}^r f$ is r-minclean by Theorem 7.26. But, also from Theorem 7.26, $\psi_r f$ is r-maxclean. So $\mathcal{M}^r f = \gamma_r \psi_r f$ is r-maxclean by Theorem 7.28(f). So $\mathcal{M}^r f$ is r-clean, as required. That $\mathcal{N}^r f$ is r-clean is proved similarly. □

In the same way that Theorem 7.25 and Theorem 7.26 give Theorem 7.27, Theorem 7.25(c) and Theorem 7.31 give the following result.

Theorem 7.32 (Idempotence of $\mathcal{M}$, $\mathcal{N}$) *For each integer, $r \geq 1$, the operators, $\mathcal{M}^r$ and $\mathcal{N}^r$ are idempotent; i.e.*

$$\mathcal{M}^r\mathcal{M}^r = \mathcal{M}^r, \qquad \textit{and} \qquad \mathcal{N}^r\mathcal{N}^r = \mathcal{N}^r.$$

This, like Theorem 7.27, is a generally useful result that is well known in the case of rigid structuring elements (windows) but less well known for connected sets[42].

7.3 The M- and N-Sieves

Finally we can now define M- and N-sieves of a function, $f \in \mathbf{Z}^V$, Definition 7.33

Definition 7.33 *Suppose that $f \in \mathbf{Z}^V$.*

(a) The M-sieve of f is the sequence, $(f_r)_{r=1}^{\infty}$, given by

$$f_1 = \mathcal{M}^1 f = f, \qquad f_{r+1} = \mathcal{M}^{r+1} f_r,$$

for integers, $r \geq 1$.

(b) The N-sieve of f is the sequence, $(f_r)_{r=1}^{\infty}$, given by

$$f_1 = \mathcal{N}^1 f = f, \qquad f_{r+1} = \mathcal{N}^{r+1} f_r,$$

for integers, $r \geq 1$.

So from Theorem 7.31 we have Theorem 7.34.

The operators, $(M^r)_{r=1}^{\infty}$ and $(N^r)_{r=1}^{\infty}$, as defined in Definition 7.35.

Theorem 7.34 *If $(f_r)_{r=1}^{\infty}$ is the M- or the N-sieve [3] of an $f \in \mathbf{Z}^V$, then, for each integer, $r \geq 1$, f_r is r-clean.*

One important property of the M- and N-sieves is that, for a fixed $\{x, y\} \in E$, as r increases, the jump, $f_r(y) - f_r(x)$, does not change sign and its absolute value never increases. In particular, if it vanishes for some r, it is then zero for all larger r. This is stated formally in Theorem 7.36.

[3] For notational convenience we also define the operators, $(M^r)_{r=1}^{\infty}$ and $(N^r)_{r=1}^{\infty}$, as follows.

Definition 7.35 *For each integer, $r \geq 1$, we let*

$$M^r = \mathcal{M}^r \cdots \mathcal{M}^2\mathcal{M}^1, \qquad \textit{and} \qquad N^r = \mathcal{N}^r \cdots \mathcal{N}^2\mathcal{N}^1.$$

So the M-sieve (resp. N-sieve) of an $f \in \mathbf{Z}^V$ is $(M^r f)_{r=1}^{\infty}$ (resp. $(N^r f)_{r=1}^{\infty}$).

Theorem 7.36 (Causality) *Suppose that $(f_r)_{r=1}^{\infty}$ is the M- or the N-sieve of an $f \in \mathbf{Z}^V$, that $\{x, y\} \in E$ and put $\delta_r = f_r(y) - f_r(x)$, for each, r. Then*

$$\delta_1 \geq \delta_2 \geq \cdots \geq 0, \qquad \text{or} \qquad \delta_1 \leq \delta_2 \leq \cdots \leq 0.$$

In other words n-dimensional sieves preserve scale-space causality.

Proof: By interchanging x and y if necessary, we can assume that $f(x) \leq f(y)$. Since $f_{r+1} = \mathcal{M}^r f_r$ or $f_{r+1} = \mathcal{N}^r f_r$, the result follows from Theorem 7.37 below. □

Theorem 7.37 *If $f \in \mathbf{Z}^V$ and $\{x, y\} \in E$ with $f(x) \leq f(y)$, then, for each integer, $r \geq 1$,*

$$0 \leq \mathcal{M}^r f(y) - \mathcal{M}^r f(x) \leq f(y) - f(x), \tag{10}$$

$$0 \leq \mathcal{N}^r f(y) - \mathcal{N}^r f(x) \leq f(y) - f(x). \tag{11}$$

Proof: From Theorem 7.28(a) with $g = \mathcal{M}^r f$ and $g = f - \mathcal{M}^r f$, we obtain that $\mathcal{M}^r f(x) \leq \mathcal{M}^r f(y)$, and $f(x) - \mathcal{M}^r f(x) \leq f(y) - \mathcal{M}^r f(y)$. The first gives that $0 \leq \mathcal{M}^r f(y) - \mathcal{M}^r f(x)$ and the second rearranges to give that $\mathcal{M}^r f(y) - \mathcal{M}^r f(x) \leq f(y) - f(x)$. So (10) is proved. (11) is proved in a similar manner. □

In the M-sieve, f_{r+1} is computed by applying ψ_{r+1} to the r-clean function, f_r, and then applying γ_{r+1} to the resulting function, $\psi_{r+1} f_r$, which is r-clean by Theorem 7.28(f). So the following result is useful in computing f_{r+1} from f_r.

Theorem 7.38 *For each integer, $r \geq 1$, we have the following: (a) Suppose that $f \in \mathbf{Z}^V$ is r-maxclean and let $\mathcal{C}$ be the set of all maximum cells of f with r elements. Then, for each $x \in V$,*

$$\psi_{r+1} f(x) = \begin{cases} \max_{u \in \operatorname{nbr} C} f(u), & \text{if } x \in C \in \mathcal{C}, \\ f(x), & \text{if } x \notin \bigcup \mathcal{C}. \end{cases} \tag{12}$$

(b) Suppose that $f \in \mathbf{Z}^V$ is r-minclean and let $\mathcal{C}$ be the set of all minimum cells of f with r elements. Then, for each $x \in V$,

$$\gamma_{r+1} f(x) = \begin{cases} \min_{u \in \operatorname{nbr} C} f(u), & \text{if } x \in C \in \mathcal{C}, \\ f(x), & \text{if } x \notin \bigcup \mathcal{C}. \end{cases} \tag{13}$$

Note that, in (12) and (13), $\operatorname{nbr} C \neq \emptyset$ by Theorem 7.8 since $|C| = r < |V|$. So the maximum and minimum make sense.

Proof of Theorem 7.38: Since (b) will follow from (a) applied to $-f$, we shall only consider (a).

Suppose that $x \in C \in \mathcal{C}$. By Definition 7.15, let $\xi \in \mathcal{C}_{r+1}(G, x)$ be such that

$$\min_{u\in\xi} f(u) = \psi_{r+1} f(x).$$

Then $\xi \not\subseteq C$ since $|\xi| > |C|$. So, by Theorem 7.7, $\xi \cap \operatorname{nbr} C \neq \emptyset$. Let $v \in \xi \cap \operatorname{nbr} C$. Then

$$\psi_{r+1} f(x) = \min_{u\in\xi} f(u) \leq f(v) \leq \max_{u\in\operatorname{nbr} C} f(u). \tag{14}$$

Now, for each $u \in \operatorname{nbr} C$, we have that $C \cup \{u\} \in \mathcal{C}_{r+1}(G, x)$ and that $f(u) < f(C)$ (C is a maximum cell of f). So, by Definition 7.15,

$$\max_{u\in\operatorname{nbr} C} f(u) = \max_{u\in\operatorname{nbr} C} \min_{w\in C\cup\{u\}} f(u) \leq \psi_{r+1} f(x). \tag{15}$$

Together (14) and (15) give (12) in the case when $x \in C \in \mathcal{C}$.

Now suppose that $x \in V \setminus \bigcup \mathcal{C}$. Let

$$A = \{\, u \in V \mid f(u) \geq f(x) \,\},$$

and let C be the connected component of A containing x. Then, as in the proof of Theorem 7.25, $\operatorname{ent} C \subseteq \operatorname{inc} f$.

We now show that $|C| \geq r + 1$. Suppose, for a contradiction, that $|C| \leq r$. Then, since $\operatorname{ent} C \subseteq \operatorname{inc} f$, Lemma 7.24 gives that there exits a maximum cell, D, of f with $D \subseteq C$. Since f is r-maxclean, D must have at least r elements and so, since $|C| \leq r$, D must be equal to C and have precisely r elements. So C is a maximum cell of f with r elements; i.e. $C \in \mathcal{C}$. This contradicts that $x \in C$ and $x \in V \setminus \bigcup \mathcal{C}$. So we have proved that $|C| \geq r + 1$.

By Theorem 7.10, there exists $\xi \in \mathcal{C}_{r+1}(G, x)$ with $\xi \subseteq C \subseteq A$. Thus, by Definition 7.15,

$$\psi_{r+1} f(x) \geq \min_{u\in\xi} f(u) \geq f(x).$$

So with Theorem 7.16(c), we obtain that $\psi_{r+1} f(x) = f(x)$. This proves (12) in the case when $x \in V \setminus \bigcup \mathcal{C}$. $\square$

7.4 The M-Granule Decomposition

When we have the M-sieve, (f_r), of a function, $f \in \mathbf{Z}^V$, we can consider the differences, $f_r - f_{r+1}$ and define granules as in Definition 7.39.

Definition 7.39 (Sieve decomposition) *When $f \in \mathbf{Z}^V$, the M-granule decomposition of f is the sequence of functions, $(d_r)_{r=1}^{\infty}$, defined by*

$$d_r = f_r - f_{r+1}, \qquad \textit{for each integer, } r \geq 1. \tag{16}$$

Similarly the N-granule decomposition of f can be defined.

In this section we will give necessary and sufficient conditions for a sequence $(d_r)_{r=1}^{\infty} \subseteq \mathbf{Z}^V$ to be an M-granule decomposition of some $f \in \mathbf{Z}^V$. As we shall see, one of the conditions is that, for each r, d_r is an r^{th}-order M-granule function, where we make the following definition.

Definition 7.40 *For each integer, $r \geq 1$, an r^{th}-order M-granule function is a $d \in \mathbf{Z}^V$ such that*[4]

(i) each cell, C, of d with $d(C) \neq 0$ has r elements,

(ii) $d(x) \leq 0$, for all $x \in \operatorname{nbr} C$, for each cell, C, of d with $d(C) > 0$,

(iii) $d(x) \geq 0$, for all $x \in \operatorname{nbr} C$, for each cell, C, of d with $d(C) < 0$,

(iv) each connected component of $[d \leq 0]$ has at least r elements,

(v) each connected component of $[d \geq 0]$ has at least $r+1$ elements.

When d is an r^{th}-order M-granule function, the cells, C, of d with $d(C) \neq 0$ are known as the granules of d. The granules, C, of d with $d(C) > 0$ (resp. $d(C) < 0$) are known as the positive (resp. negative) granules of d.

The following result gives some elementary properties of M-granule functions.

[4] When $f_1, \ldots, f_n \in \mathbf{Z}^V$ and P is an n place predicate on $\mathbf{Z}$, $[P(f_1, \ldots, f_n)]$ will be used to denote the set $\{ x \in V \mid P(f_1(x), \ldots, f_n(x)) \}$. So, for example, $[d \geq 0]$ denotes the set $\{ x \in V \mid d(x) \geq 0 \}$.

Theorem 7.41 *If d is an r^{th}-order M-granule function for some integer, $r \geq 1$, then we have the following.*[5]

(a) d *is* r*-clean and* d^- *is* $(r+1)$*-minclean.*

(b) $\operatorname{inc} d^+ = \operatorname{ent}[d > 0] = \bigcup \{ \operatorname{ent} C \mid C$ *is a positive granule of* $d \}$.

(c) $\operatorname{inc} d^- = \operatorname{ent}[d < 0] = \bigcup \{ \operatorname{ent} C \mid C$ *is a negative granule of* $d \}$.

(d) $\operatorname{inc} d = \operatorname{inc} d^+ \cup \operatorname{dec} d^- = \operatorname{ent}[d > 0] \cup \operatorname{lve}[d < 0]$.

Proof: (a): For the first part we have to show that all extremal cells of d have at least r elements. Suppose then that C is an extremal cell of d. Then, if $d(C) \neq 0$, we have that $|C| = r$ by Definition 7.40(i). So we may assume that $d(C) = 0$.

If C is a maximum cell of d, then $d(x) < d(C) = 0$, for all $x \in \operatorname{nbr} C$. So C is a connected component of $[d \geq 0]$ and so $|C| \geq r+1$ by Definition 7.40(v).

Similarly, if C is a minimum cell of d, then C is a connected component of $[d \leq 0]$ and so $|C| \geq r$ by Definition 7.40(iv).

For the second part we have to show that all minimum cells of d^- have at least $r+1$ elements.

For each negative granule, D, of d, we have that $|D| = r < |V|$. So, by Theorem 7.8, $\operatorname{nbr} D \neq \emptyset$. By Definition 7.40(iii), $d^-(x) = 0 < d^-(D)$, for all $x \in \operatorname{nbr} D$. So D is not a minimum cell of d^-.

Thus, for each minimum cell, C, of d^-, C is not a negative granule of d and so $d^-(C) = 0$. So C is a connected component of $[d^- = 0] = [d \geq 0]$. So, by Definition 7.40(v), $|C| \geq r+1$, as required.

(b): Denote the union by H. We show that

$$\operatorname{inc} d^+ \subseteq H \subseteq \operatorname{ent}[d > 0] \subseteq \operatorname{inc} d^+. \tag{17}$$

If $(x, y) \in \operatorname{inc} d^+$, then $d^+(y) > d^+(x) \geq 0$. So $d(y) > 0$ and so y belongs to a positive granule, C, of d. But, as $d(x) \neq d(y)$, we have that $x \notin C$ and so $x \in \operatorname{nbr} C$. So $(x, y) \in \operatorname{ent} C \subseteq H$.

[5] When $t \in \mathbf{Z}$, we define $t^+ = \max(t, 0)$ and $t^- = \max(-t, 0)$. For $f \in \mathbf{Z}^V$, f^+ and f^- are defined pointwise.

If (x,y) is any element of H, then $(x,y) \in \operatorname{ent} C$, for some positive granule C of d. Since $x \in \operatorname{nbr} C$, Definition 7.40(ii) gives that $d(x) \leq 0$; i.e. $x \notin [d>0]$. Since $y \in C \subseteq [d>0]$, we have that $(x,y) \in \operatorname{ent}[d>0]$.

If (x,y) is any element of $\operatorname{ent}[d>0]$, then $d(x) \leq 0 < d(y)$ since $x \notin [d>0]$ and $y \in [d>0]$. So $d^+(x) = 0 < d^-(y)$ and so $(x,y) \in \operatorname{inc} d^+$. This completes the proof of (17) and hence of (b).

(c): This follows in a similar manner to (b).

(d): The second equality follows from (b) and (c). So we only need to consider the first equality. By Theorem 7.22 with $n=2$, $f_1 = d^+$ and $f_2 = -d^-$, we have that

$$\operatorname{inc} d \subseteq \operatorname{inc} d^+ \cup \operatorname{dec} d^-.$$

For the reverse inclusion, we note that $\{x,y\} \in E,\ d(x) \leq d(y) \ \Rightarrow\ d^+(x) \leq d^+(y)$ and $d^-(x) \leq d^-(y)$.

So, by Lemma 7.30(c), $\operatorname{inc} d^+ \subseteq \operatorname{inc} d$ and $\operatorname{inc} d^- \subseteq \operatorname{inc} d$. Thus we have proved (d). $\square$

The following result consider a single step in the M-granule decomposition.

Theorem 7.42 *If $f \in \mathbf{Z}^V$ is r-clean, for some integer, $r \geq 1$, and we write,*

$$f = g + d, \qquad \textit{where} \quad g = \mathcal{M}^{r+1} f,$$

then

(i) d *is an* r^{th}*-order* M*-granule function,*

(ii) g *is* $(r+1)$*-clean,*

(iii) g *is constant on each granule of* d*,*

(iv) $\operatorname{inc} d \cap \operatorname{dec} g = \emptyset$*,*

(v) $\emptyset \neq A \subseteq V,\ \operatorname{ent} A \subseteq \operatorname{inc} g \cup \operatorname{dec} d^- \ \Rightarrow\ |A| \geq r+1.$

We also have that

$$\operatorname{inc} f = \operatorname{inc} g \cup \operatorname{inc} d. \tag{18}$$

The proof of Theorem 7.42 is rather technical and is deferred to §7.5 below. It can be read now if desired.

As a consequence of Theorem 7.42, we have the following result.

Theorem 7.43 *For each $f \in \mathbf{Z}^V$, the M-granule decomposition, $(d_r)_{r=1}^{\infty}$, of f satisfies the following.*

(i) d_r *is an* r^{th}*-order M-granule function, for each integer, $r \geq 1$.*

(ii) d_s *is constant on each granule of d_r, whenever $s > r \geq 1$.*

(iii) $\operatorname{inc} d_r \cap \operatorname{dec} d_s = \emptyset$, *for all integers, $r, s \geq 1$.*

Furthermore, if $(f_r)_{r=1}^{\infty}$ denotes the M-sieve of f, then

$$\operatorname{inc} f_r = \operatorname{inc} d_r \cup \operatorname{inc} f_{r+1}, \tag{19}$$

for each integer, $r \geq 1$.

Proof: Let $(f_r)_{r=1}^{\infty}$ be the M-sieve of f. Then, by Theorem 7.34, Theorem 7.42 applies with (f_r, f_{r+1}, d_r) as (f, g, d). This immediately gives (i) and (19). This leaves (ii) and (iii).

(ii): Fix an integer, $r \geq 1$ and suppose that C is a granule of d_r. If f_s is constant C, then $f_{s+1} = \mathcal{M}^{s+1} f_s$ is constant on C by Theorem 7.28(c). So, since f_{r+1} is constant C by Theorem 7.42(iii), it follows, by induction on s, that f_s is constant on C, for all integers, $s > r$. So $d_s = f_s - f_{s+1}$ is constant on C, for all integers, $s > r$.

(iii): Fix an integer, $r \geq 1$. Then, for each integer, $s > r$, $\operatorname{dec} f_s = \operatorname{dec} d_s \cup \operatorname{dec} f_{s+1}$ by (19). So

$$\begin{aligned} &\operatorname{inc} d_r \cap \operatorname{dec} f_s = \emptyset \\ &\qquad \Rightarrow \begin{cases} \operatorname{inc} d_r \cap \operatorname{dec} d_s = \emptyset, \\ \operatorname{inc} d_r \cap \operatorname{dec} f_{s+1} = \emptyset. \end{cases} \end{aligned} \tag{20}$$

So, since $\operatorname{inc} d_r \cap \operatorname{dec} f_{r+1} = \emptyset$ by Theorem 7.42(iv), we obtain, by induction on s, that $\operatorname{inc} d_r \cap \operatorname{dec} f_s = \emptyset$, for all integers, $s > r$. So, by (20), $\operatorname{inc} d_r \cap \operatorname{dec} d_s = \emptyset$, for all integers, $s > r$. Thus we have proved (iii) in the case when $r < s$.

The case when $r = s$ is obvious. The case when $r > s$ follows by taking the transpose of the $r < s$ case and interchanging the roles of r and s. $\square$

In view of Theorem 7.43(i), we can make the following definition.

Definition 7.44 *When $(d_r)_{r=1}^{\infty}$ is the M-granule decomposition of an $f \in \mathbf{Z}^V$, the granules of d_r are known as the r^{th}-order M-granules of f. The positive* *(resp. negative) granules of d_r are known as the r^{th}-order positive (resp. negative) M-granules of f.*

In order to simplify matters we shall consider the M-granule decomposition of functions with finite support, where we make the following definition.

Definition 7.45 *For an $f \in \mathbf{Z}^V$, the set, $[f \neq 0]$, is known as the support of f. We say that $f \in \mathbf{Z}^V$ has finite support if and only if its support, $[f \neq 0]$, is a finite set.*

Theorem 7.46 *If $(f_r)_{r=1}^{\infty}$ is the M-sieve and $(d_r)_{r=1}^{\infty}$ the M-granule decomposition of an $f \in \mathbf{Z}^V$ with finite support, then*

(i) $f_r = d_r = 0$ on $V \setminus A$, for all integers, $r \geq 1$, for some finite subset, A, of V,

(ii) $f_r = d_r = 0$, for all integers, $r \geq r_0$, for some integer, $r_0 \geq 1$.

Proof: Since $d_r = f_r - f_{r+1}$, the results for d_r follow from the results for f_r. So we need only consider f_r.

(i): By Theorem 7.14, $[f = 0] = V \setminus [f \neq 0]$ has finitely many connected components. Let $\mathcal{C}$ be finite set of all the connected components of $[f = 0]$ and let $\mathcal{C}_1 = \{\, C \in \mathcal{C} \mid |C| < \infty \,\}$, $\mathcal{C}_\infty = \{\, C \in \mathcal{C} \mid |C| = \infty \,\}$.

Put $A = [f \neq 0] \cup \bigcup \mathcal{C}_1$. Then A is a finite subset of V and $V \setminus A = \bigcup \mathcal{C}_\infty$.

Next we show that for each integer, $r \geq 1$, and each $h \in \mathbf{Z}^V$,

$$h = 0 \text{ on } V \setminus A \;\Rightarrow\; \psi_r h = \gamma_r h = 0 \text{ on } V \setminus A. \tag{21}$$

So suppose that the right hand side of (21) holds and suppose that $x \in V \setminus A$. Then $x \in C$, for some $C \in \mathcal{C}_\infty$. By Theorem 7.10, there exists a $\xi \in \mathcal{C}_r(x, G)$ with $\xi \subseteq C$. So, by Definition 7.15 and Theorem 7.16(c),

$$\begin{aligned} 0 &= \min_{u \in \xi} h(u) \\ &\leq \psi_r h(x) \leq h(x) \leq \gamma_r h(x) \leq \max_{u \in \xi} h(u) \\ &= 0. \end{aligned}$$

This gives that $\psi_r h(x) = \gamma_r h(x) = 0$, as required.

Now if $f_r = 0$ on $V \setminus A$, for some integer, $r \geq 1$, then, by (21), $\psi_{r+1} f_r = 0$ on $V \setminus A$ and so $f_{r+1} = \gamma_{r+1}\psi_{r+1} f_r = 0$ on $V \setminus A$. So, since $f_1 = f = 0$ on $V \setminus A$, (i) for f_r now follows by induction on r.

(ii): Let $r_0 = |A| + 1$ and suppose that $r \geq r_0$ is an integer.

Take $x \in A$ to be so that $f_r(x)$ is greatest and let C be the cell of f_r containing x. Then, by (i), if $f_r(x) > 0$, we can see that C is a maximum cell of f_r and $C \subseteq A$. Since $|C| \leq |A| \leq r_0 < r$, this contradicts that f_r is r-clean. So $f_r(x) \leq 0$ and so $f_r \leq 0$ on A.

Similarly, by taking $y \in A$ to be so that $f_r(y)$ is least and by showing that the cell of f_r containing y is a minimum cell of f_r if $f_r(y) < 0$, we see that $f_r \geq 0$ on A.

So, with (ii), we have that $f_r = 0$, for all integers, $r \geq r_0$, as required. □

We can now give the promised characterization of the M-granule decomposition of functions of finite support.

Theorem 7.47 (Inversion) *A sequence $(d_r)_{r=1}^{\infty} \subseteq \mathbf{Z}^V$ is the M-granule decomposition of a function in $\mathbf{Z}^V$ with finite support if and only if*

(i) $d_r = 0$, for all integers, $r \geq r_0$, for some integer, $r_0 \geq 1$,

(ii) d_r is an r^{th}-order M-granule function with finite support, for each integer, $r \geq 1$,

(iii) d_s is constant on each granule of d_r, whenever $s > r \geq 1$,

(iv) $\operatorname{inc} d_r \cap \operatorname{dec} d_s = \emptyset$, for all integers, $r, s \geq 1$,

(v) $r \geq 2$, $\emptyset \neq A \subseteq V$, $\operatorname{ent} A \subseteq (\bigcup_{s=r}^{\infty} \operatorname{inc} d_s) \cup \operatorname{dec} d_{r-1}^{-} \Rightarrow |A| \geq r$,

(vi) $r \geq 1$, $\emptyset \neq A \subseteq V$, $\operatorname{ent} A \subseteq \bigcup_{s=r}^{\infty} \operatorname{dec} d_s \Rightarrow |A| \geq r$,

Moreover, if $(d_r)_{r=1}^{\infty} \subseteq \mathbf{Z}^V$ is the M-granule decomposition of an $f \in \mathbf{Z}^V$ with finite support, then

$$f = \sum_{r=1}^{\infty} d_r, \quad \textit{and} \quad \operatorname{inc} f = \bigcup_{r=1}^{\infty} \operatorname{inc} d_r. \tag{22}$$

If further, $(f_r)_{r=1}^{\infty}$ is the M-sieve of f, then, for each integer, $r \geq 1$,

$$f_r = \sum_{s=r}^{\infty} d_s, \quad \textit{and} \quad \operatorname{inc} f_r = \bigcup_{s=r}^{\infty} \operatorname{inc} d_s. \tag{23}$$

Note that, by (i), each of the 'infinite' sums in Theorem 7.47 is in fact a finite sum.

To prove the 'if' part of Theorem 7.47, we shall need the following partial converse to Theorem 7.42. Its proof, like that of Theorem 7.42, is deferred to §7.5 below. It can be read now if desired.

Theorem 7.48 *If, for some integer, $r \geq 1$, $f = g + d$, where $g, d \in \mathbf{Z}^V$ are such that*

(i) d *is an* r^{th}*-order* M*-granule function,*

(ii) g *is* $(r+1)$*-clean,*

(iii) g *is constant on each granule of* d*,*

(iv) $\operatorname{inc} d \cap \operatorname{dec} g = \emptyset$*,*

(v) $\emptyset \neq A \subseteq V$*,* $\operatorname{ent} A \subseteq \operatorname{inc} g \cup \operatorname{dec} d^- \;\Rightarrow\; |A| \geq r + 1$*,*

then $\mathcal{M}^{r+1} f = g$*.*

Proof of Theorem 7.47: *'only if' part*: Suppose that $(d_r)_{r=1}^{\infty} \subseteq \mathbf{Z}^V$ is the M-granule decomposition of an $f \in \mathbf{Z}^V$ with finite support. Then (i) follows from Theorem 7.46(ii) and each d_r has finite support by Theorem 7.46(i). (ii)–(iv) now follow directly from Theorem 7.43. This leaves (v) and (vi). Before proving these we establish (22) and (23).

Let $(f_r)_{r=1}^{\infty}$ be the M-sieve of f. For each integer, $s \geq 1$, $f_s = d_s + f_{s+1}$ and, by (19) of Theorem 7.43, $\operatorname{inc} f_s = \operatorname{inc} d_s \cup \operatorname{inc} f_{s+1}$. So, for each integer $r \geq 1$, we have, by induction on t, that for each integer, $t > r$, $f_r = \sum_{s=r}^{t-1} d_s + f_t$, and $\operatorname{inc} f_r = \left(\bigcup_{s=r}^{t-1} \operatorname{inc} d_s \right) \cup \operatorname{inc} f_t$.

So, by Theorem 7.46(ii), taking t to be sufficiently large gives (23). Since $f_1 = f$, taking $r = 1$ in (23) gives (22).

(v): Suppose that $r \geq 2$. Then, by Theorem 7.42(v), with $(r-1, f_{r-1}, f_r, d_{r-1})$ as (r, f, g, d), $\emptyset \neq A \subseteq V$, $\operatorname{ent} A \subseteq \operatorname{inc} f_r \cup \operatorname{dec} d_{r-1}^- \;\Rightarrow\; |A| \geq r$. Substituting the value of $\operatorname{inc} f_r$ given by (23) gives (v).

(vi): Suppose that $r \geq 1$. Then, since f_r is r-minclean, Theorem 7.23(b) gives that $\emptyset \neq A \subseteq V$, $\operatorname{ent} A \subseteq \operatorname{dec} f_r \;\Rightarrow\; |A| \geq r$.

Substituting the value of $\operatorname{dec} f_r$ given by (23) gives (vi).

'if' part: Suppose that $(d_r)_{r=1}^{\infty} \subseteq \mathbf{Z}^V$ satisfy (i)–(vi) and put

$$f = \sum_{r=1}^{\infty} d_r, \quad \text{and} \quad f_r = \sum_{s=r}^{\infty} d_s, \tag{24}$$

for each $r \geq 1$. Clearly from (i) and (ii), f has finite support. From Theorem 7.22 and (i), we have, for each integer, $r \geq 1$, that

$$\operatorname{inc} f_r \subseteq \bigcup_{s=r}^{\infty} \operatorname{inc} d_s. \tag{25}$$

From (v), (vi) and (25), we see that

$$\begin{aligned} &r \geq 2, \emptyset \neq A \subseteq V, \\ &\quad \operatorname{ent} A \subseteq \operatorname{inc} f_r \cup \operatorname{dec} d_{r-1}^{-} \;\Rightarrow\; |A| \geq r, \end{aligned} \tag{26}$$

$$\begin{aligned} &r \geq 1, \emptyset \neq A \subseteq V, \\ &\quad \operatorname{ent} A \subseteq \operatorname{dec} f_r \;\Rightarrow\; |A| \geq r. \end{aligned} \tag{27}$$

From (27) and Theorem 7.23(b), f_r is r-minclean, for each r. Since (26) holds when $\operatorname{inc} f_r \cup \operatorname{dec} d_{r-1}^{-}$ is replaced by its subset, $\operatorname{inc} f_r$, (26) and Theorem 7.23(b) give that f_r is r-maxclean for each r (for $r = 1$, we note that all functions in $\mathbf{Z}^V$ are 1-maxclean). Thus we have proved that

$$f_r \text{ is } r\text{-clean}, \qquad \text{for each } r \geq 1. \tag{28}$$

From (iii) and (24),

$$\begin{aligned} &f_{r+1} \text{ is constant on each granule of } d_r, \\ &\qquad\qquad \text{for each } r \geq 1, \end{aligned} \tag{29}$$

and from (iv) and the transpose of (25),

$$\operatorname{inc} d_r \cap \operatorname{dec} f_{r+1} = \emptyset, \qquad \text{for each } r \geq 1. \tag{30}$$

By (ii), (28), (29), (30) and (26), Theorem 7.48 applies with (f_r, f_{r+1}, d_r) as (f, g, d) to give that

$$\mathcal{M}^{r+1} f_{r+1} = f_r, \qquad \text{for each } r \geq 1.$$

Since $f_1 = f$, we see that $(f_r)_{r=1}^{\infty}$ is the M-sieve of f. So, since $f_{r+1} - f_r = d_r$, $(d_r)_{r=1}^{\infty}$ is the M-granule decomposition of f, as required. $\square$

From Theorem 7.47, we see that the M-granule decomposition has the important invertibility property, Theorem 7.49.

Theorem 7.49 (Invertibility) *Suppose that $(d_r)_{r=1}^{\infty}$ is the M-granule decomposition of an $f \in \mathbf{Z}^V$ with finite support then the original image can be rebuilt from the granularity domain*

$$f = \sum_{r=1}^{\infty} d_r. \tag{31}$$

Suppose that $(\hat{d}_r)_{r=1}^{\infty} \subseteq \mathbf{Z}^V$ are such that, for each integer, $r \geq 1$, $\hat{d}_r = 0$ on $[d_r = 0]$, $\hat{d}_r$ is constant and nonnegative on each positive granule of d_r and $\hat{d}_r$ is constant and nonpositive on each negative granule of d_r and let

$$\hat{f} = \sum_{r=1}^{\infty} \hat{d}_r. \tag{32}$$

Then $(\hat{d}_r)_{r=1}^{\infty}$ is the M-granule decomposition of $\hat{f}$.

Proof: It is clear, for each $r \geq 1$, that $\hat{d}_r$ is an r^{th}-order M-granule function and that the set of all positive (resp. negative) granules of $\hat{d}_r$ is a subset of the set of all positive (resp. negative) granules of d_r. Thus, in particular, Theorem 7.41(b) and (c) give that

$$\operatorname{inc} \hat{d}_r^{+} \subseteq \operatorname{inc} d_r^{+}, \quad \text{and} \quad \operatorname{inc} \hat{d}_r^{-} \subseteq \operatorname{inc} d_r^{-}. \tag{33}$$

So, by Theorem 7.41(d),

$$\operatorname{inc} \hat{d}_r \subseteq \operatorname{inc} d_r. \tag{34}$$

Now, by Theorem 7.47, $(d_r)_{r=1}^{\infty}$ satisfies (i)–(vi) of Theorem 7.47. So, by (33), (34) and that $\hat{d}_r$ is constant on any connected subset of V that d_r is constant on, we see that $(\hat{d}_r)_{r=1}^{\infty}$ satisfies (i)–(vi) of Theorem 7.47. Thus, by Theorem 7.47, $(\hat{d}_r)_{r=1}^{\infty}$ is the M-granule decomposition of some function in $\mathbf{Z}^V$ with finite support. By (22) of Theorem 7.47, this function is $\hat{f}$. This completes the proof. $\square$

7.5 The Deferred Proofs

To prove Theorem 7.42, we use the following lemma.

Lemma 7.50 *If $f \in \mathbf{Z}^V$ is r-maxclean, for some integer, $r \geq 1$, and we write,*

$$f = h + d, \qquad \text{where} \quad h = \psi_{r+1} f,$$

then $d \geq 0$ and, for each cell, C, of d with $d(C) \neq 0$, we have that

(i) $|C| = r$,

(ii) $d(x) = 0$, *for all* $x \in \operatorname{nbr} C$,

(iii) h *is constant on* $C \cup \{x\}$, *for some* $x \in \operatorname{nbr} C$,

(iv) $h(x) \leq h(C)$, *for all* $x \in \operatorname{nbr} C$.

Proof: We have that $d \geq 0$ by Theorem 7.16(c).

Now let $\mathcal{C}$ be the set of all maximum cells of f with r elements. We shall show that (i)–(iv) hold, for each $C \in \mathcal{C}$. So suppose that $C \in \mathcal{C}$. Then C clearly satisfies (i).

By Theorem 7.38(a),

$$h \text{ is constant on } C, \quad \text{and}$$

$$h(C) = \max_{u \in \operatorname{nbr} C} f(u). \tag{35}$$

Since C is a maximum cell of f, no element of $\operatorname{nbr} C$ can belong to a maximum cell of f. So $\operatorname{nbr} C \subseteq V \setminus \bigcup \mathcal{C}$ and so, by Theorem 7.38(a),

$$h(x) = f(x), \qquad \text{for all } x \in \operatorname{nbr} C. \tag{36}$$

Since $d = f - h$, (36) gives that C satisfies (ii).

For each $x \in \operatorname{nbr} C$, (35) and (36) give that $h(x) = f(x) \leq h(C)$. So C satisfies (iv).

If we take $x \in \operatorname{nbr} C$ so that $f(x) = \max_{u \in \operatorname{nbr} C} f(u)$, then (35) and (36) give that $h(x) = f(x) = h(C)$. So h is constant on $C \cup \{x\}$. Thus C satisfies (iii). This completes the proof that (i)–(iv) hold, for all $C \in \mathcal{C}$.

For each $C \in \mathcal{C}$, we have, since C is a maximum cell of f, that (35) gives that $h(C) < f(C)$. So $d = f - h$ is constant on C with $d(C) > 0$. Thus, with (ii), we see that each $C \in \mathcal{C}$ is a cell of d with $d(C) \neq 0$.

Now suppose that D is a cell of d that does not belong $\mathcal{C}$. Then, since all elements of $\mathcal{C}$ are cells of d, D must be disjoint from each element of $\mathcal{C}$. So

$D \subseteq V \setminus \bigcup \mathcal{C}$. Thus, by Theorem 7.38(a), $g = f$ on D and so $d(D) = 0$. So we see that all cells, C, of d with $d(C) \neq 0$ belong to $\mathcal{C}$ and so satisfy (i)–(iv), as required. □

Proof of Theorem 7.42: First we note that (18) holds since it follows from Theorem 7.28(e).

Now let $h = \psi_{r+1} f$; so $g = \gamma_{r+1} h$ and write

$$f = h + d_1, \qquad \text{and} \qquad h = g - d_2. \tag{37}$$

So

$$d = d_1 - d_2. \tag{38}$$

Now f is r-maxclean by hypothesis and $h = \psi_{r+1} f$ is r-minclean by Theorem 7.28(f). So Lemma 7.50 applies with (f, h, d_1) as (f, h, d)

and with $(-h, -g, d_2)$ as (f, h, d). So firstly

$$d_1, d_2 \geq 0. \tag{39}$$

Secondly, if C is a cell of d_1 with $d_1(C) \neq 0$, then $|C| = r$ and h is constant on $C \cup \{x\}$, for some $x \in \operatorname{nbr} C$. So, since $C \cup \{x\}$ is connected, $C \cup \{x\} \subseteq D$, for some cell, D, of h. Since $|D| \geq |C \cup \{x\}| = r + 1$, Theorem 7.38(b) gives that $g = \gamma_{r+1} h = h$ on D. So $d_2 = 0$ on D and hence on C. Thus we have shown that

$$[d_1 \neq 0] \cap [d_2 \neq 0] = \emptyset. \tag{40}$$

(38), (39) and (40) give that

$$d^+ = d_1, \qquad \text{and} \qquad d^- = d_2. \tag{41}$$

So we have that

$$\begin{aligned} f &= h + d^+, \quad \text{with} \quad h = \psi_{r+1} f, \\ h &= g - d^-, \quad \text{with} \quad g = \gamma_{r+1} h, \end{aligned} \tag{42}$$

and Lemma 7.50 applies with (f, h, d^+) as (f, h, d) and with $(-h, -g, d^-)$ as (f, h, d).

We can now prove (i)–(v).

(i): If C is a cell of d with $d(C) > 0$, then C is a cell of d^+ with $d^+(C) \neq 0$ and so, by Lemma 7.50(i) and (ii) with (f, h, d^+) as (f, h, d), $|C| = r$ and $d^+(x) = 0$, for all $x \in \operatorname{nbr} C$. So $d(x) \leq 0$, for all $x \in \operatorname{nbr} C$.

If C is a cell of d with $d(C) < 0$, then C is a cell of d^- with $d^-(C) \neq 0$ and so, by Lemma 7.50(i) and (ii) with $(-h, -g, d^-)$ as (f, h, d), $|C| = r$ and $d^-(x) = 0$, for all $x \in \operatorname{nbr} C$. So $d(x) \geq 0$, for all $x \in \operatorname{nbr} C$.

Thus d satisfies (i)–(iii) of Definition 7.40.

For (iv) of Definition 7.40, we have that $[d \leq 0] = [d^+ = 0]$. So each connected component of $[d \leq 0]$ is a minimum cell of d^+. Thus, since $d^+ = f - \psi_{r+1} f$ is r-minclean by Theorem 7.28(f), each connected component of $[d \leq 0]$ has at least r elements, as required.

Similarly, since $[d \geq 0] = [d^- = 0]$, (v) of Definition 7.40 will follow if we show that d^- is $(r+1)$-minclean. But $h = \psi_{r+1} f$ is $(r+1)$-maxclean by Theorem 7.26. So $-d^- = h - \gamma_{r+1} h$ is $(r+1)$-maxclean by Theorem 7.28(f), as required. This completes the proof that d is an r^{th}-order M-granule function.

(ii): This follows immediately from Theorem 7.31 since $h = \mathcal{M}^{r+1} f$.

(iii): If C is a positive granule, then, by Lemma 7.50(iii) with (f, h, d^+) as (f, h, d), h is constant on C. So, by Theorem 7.28(c), $g = \gamma_{r+1} h$ is constant on C. If C is a negative granule, then, by Lemma 7.50(iii) with $(-h, -g, d^-)$ as (f, h, d), g is constant on C. So g is constant on each granule of d, as required.

(iv): Suppose that $(x, y) \in \operatorname{inc} d$. Then, by Theorem 7.41(d), $(x, y) \in \operatorname{inc} d^+$ or $(x, y) \in \operatorname{dec} d^-$.

In the first case, Theorem 7.41(b) gives that $(x, y) \in \operatorname{ent} C$, for some positive granule, C, of d. So Lemma 7.50(iv) with (f, h, d^+) as (f, h, d) gives that $h(x) \leq h(C) = h(y)$. So, since $g = \gamma_{r+1} h$, Theorem 7.28(a) gives that $g(x) \leq g(y)$.

In the second case, Theorem 7.41(c) gives that $(x, y) \in \operatorname{lve} C$, for some negative granule, C, of d. So Lemma 7.50(iv) with $(-h, -g, d^-)$ as (f, h, d) gives that $g(y) \geq g(C) = g(x)$.

Thus we have shown, for each $(x, y) \in \operatorname{inc} d$, that $g(x) \leq g(y)$ and so $(x, y) \notin \operatorname{dec} g$, as required.

(v): Since $h = \psi_{r+1} f$ is $(r+1)$-maxclean by Theorem 7.26, Theorem 7.23(a) gives that

$$\emptyset \neq A \subseteq V,\ \operatorname{ent} A \subseteq \operatorname{inc} h \ \Rightarrow\ |A| \geq r+1. \tag{43}$$

But, Theorem 7.28(e) with (h, g) as (f, g), $\operatorname{inc} h = \operatorname{inc} g \cup \operatorname{dec} d^{-}$. Substituting this value of $\operatorname{inc} h$ into (43) gives (v). □

To prove Theorem 7.48 we use the following lemma.

Lemma 7.51 *If, for some integer, $r \geq 1$, $f = h + d$, where $h, d \in \mathbf{Z}^V$ are such that h is $(r+1)$-maxclean, $d \geq 0$ and for each cell, C, of d with $d(C) \neq 0$,*

(i) $|C| = r$,

(ii) $d(x) = 0$, *for all* $x \in \operatorname{nbr} C$,

(iii) h *is constant on* C,

(iv) $h(x) \leq h(C)$, *for all* $x \in \operatorname{nbr} C$,

then $\psi_{r+1} f = h$.

Proof: Since $h \leq f$, Theorem 7.16(d) gives that $\psi_{r+1} h \leq \psi_{r+1} f$. But $\psi_{r+1} h = h$ by Theorem 7.25(a) and $\psi_{r+1} f \leq f$ by Theorem 7.16(c). So we obtain that

$$h \leq \psi_{r+1} f \leq f = h + d. \tag{44}$$

From (44), we clearly have that $\psi_{r+1} f = h$ on $[d = 0]$.

Now we shall consider $\psi_{r+1} f$ on $[d \neq 0]$. Suppose that C is a cell of d with $d(C) \neq 0$. For $x \in \operatorname{nbr} C$, we have that $d(x) = 0$ by (ii) So $\psi_{r+1} f(x) = h(x)$ by (44). Thus, by (iii) and (iv), h is constant on C and

$$\psi_{r+1} f(x) \leq h(C), \qquad \text{for all } x \in \operatorname{nbr} C. \tag{45}$$

Now $f = h + d$ is constant with value $h(C) + d(C)$ on C. So, by Theorem 7.28(c), $\psi_{r+1} f$ is constant on C. So, by (44), $\psi_{r+1} f(C) \geq h(C)$. If, however, $\psi_{r+1} f(C) > h(C)$, then, by (45),

$$\psi_{r+1} f(x) < \psi_{r+1} f(C), \qquad \text{for all } x \in \operatorname{nbr} C.$$

So C would be a maximum cell of $\psi_{r+1} f$, which, since $|C| = r$, would contradict that, by Theorem 7.26, $\psi_{r+1} f$ is $(r+1)$-maxclean. Thus $\psi_{r+1} f(C) = h(C)$. This completes the proof that $\psi_{r+1} f = h$. □

Proof of Theorem 7.48: Put $h = g - d^-$. So we have that

$$f = h + d^+, \qquad \text{and} \qquad h = g - d^-.$$

Now suppose that C is a cell of d^- with $d^-(C) \neq 0$. Then C is a negative granule of d. So $|C| = r$ by Definition 7.40(i) and $d^-(x) = 0$, for all $x \in \operatorname{nbr} C$ by Definition 7.40(iii). g is constant on C by (iii).

Suppose that $x \in \operatorname{nbr} C$ and take $y \in C$ so that $\{x, y\} \in E$. Then, by Theorem 7.41(c) and (d), $(y, x) \in \operatorname{lve} C \subseteq \operatorname{dec} d^- \subseteq \operatorname{inc} d$. So, by (iv), $(y, x) \notin \operatorname{dec} g$. So $g(x) \geq g(y) = g(C)$. Thus Lemma 7.51 applies with $(-h, -g, d^-)$ as (f, h, d) and so gives that

$$\gamma_{r+1} h = g. \tag{46}$$

From Theorem 7.22 with $n = 2$, $f_1 = g$ and $f_2 = -d^-$, we obtain that $\operatorname{inc} h \subseteq \operatorname{inc} g \cup \operatorname{dec} d^-$. So (v) gives that

$$\emptyset \neq A \subseteq V,\ \operatorname{ent} A \subseteq \operatorname{inc} h \ \Rightarrow\ |A| \geq r + 1.$$

So, by Theorem 7.23(a), h is $(r+1)$-maxclean. Suppose that C is a cell of d^+ with $d^+(C) \neq 0$. Then C is a positive granule of d. So $|C| = r$ by Definition 7.40(i) and $d^+(x) = 0$, for all $x \in \operatorname{nbr} C$ by Definition 7.40(ii). Since g is constant on C by (iii), we have that $h = g - d^-$ is constant on C.

Suppose that $x \in \operatorname{nbr} C$ and take $y \in C$ be so that $\{x, y\} \in E$. Then, by Theorem 7.41(b) and (d), $(x, y) \in \operatorname{ent} C \subseteq \operatorname{inc} d^+ \subseteq \operatorname{inc} d$. So, by (iv), $(x, y) \notin \operatorname{dec} g$. So $g(x) \leq g(y) = g(C)$ and so, since $h = g - d^- = g$ on C and $h = g - d^- \leq g$ on V, we have that $h(x) \leq g(x) \leq h(C)$. Thus Lemma 7.51 applies with (f, h, d^+) as (f, h, d) to give that

$$\psi_{r+1} f = h. \tag{47}$$

From (46) and (47), we obtain that $\mathcal{M}^{r+1} f = g$, as required. $\square$

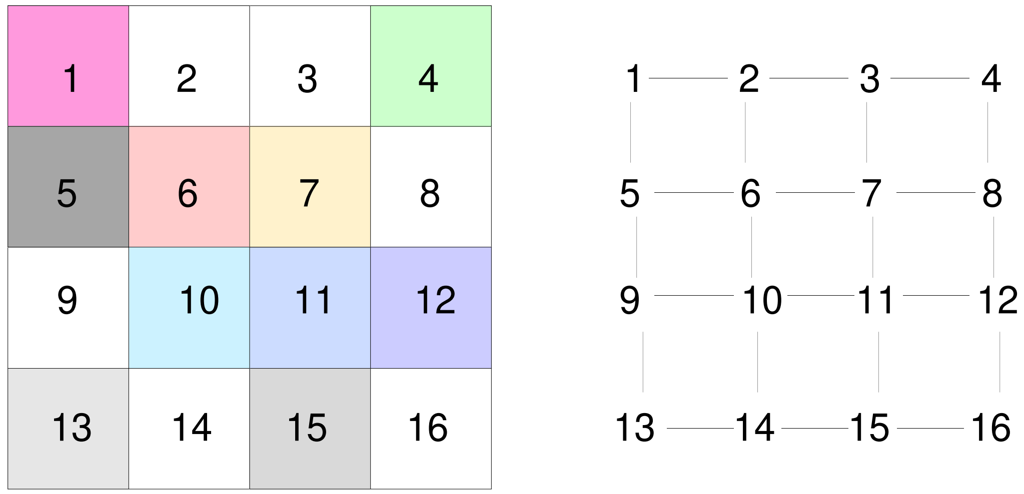

Figure 1: Graph representation of an array of four-connected pixels

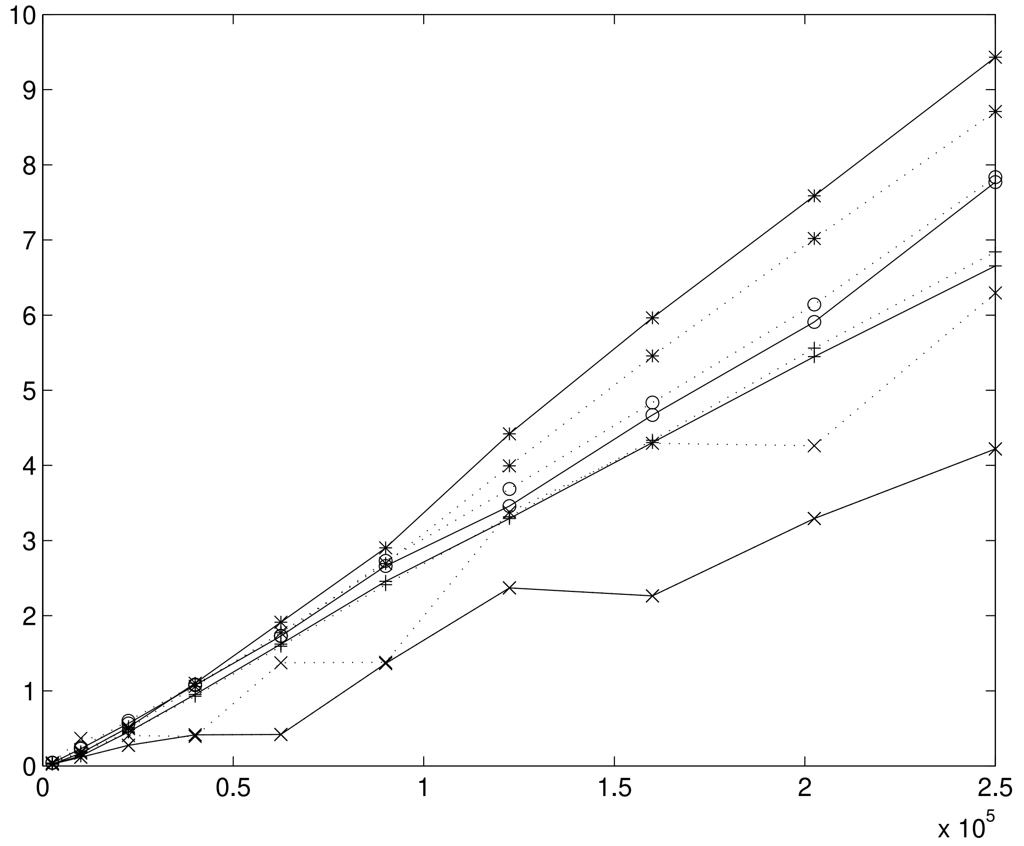

Figure 2: The order complexity of a two dimensional sieve. Solid lines are results from four images, dotted lines after random noise has been added to each image. Abscissa: area of image to be completely decomposed in pixels (0 to 500 by 500). Ordinate: computation time on Mac 7500 (0 to 10s)

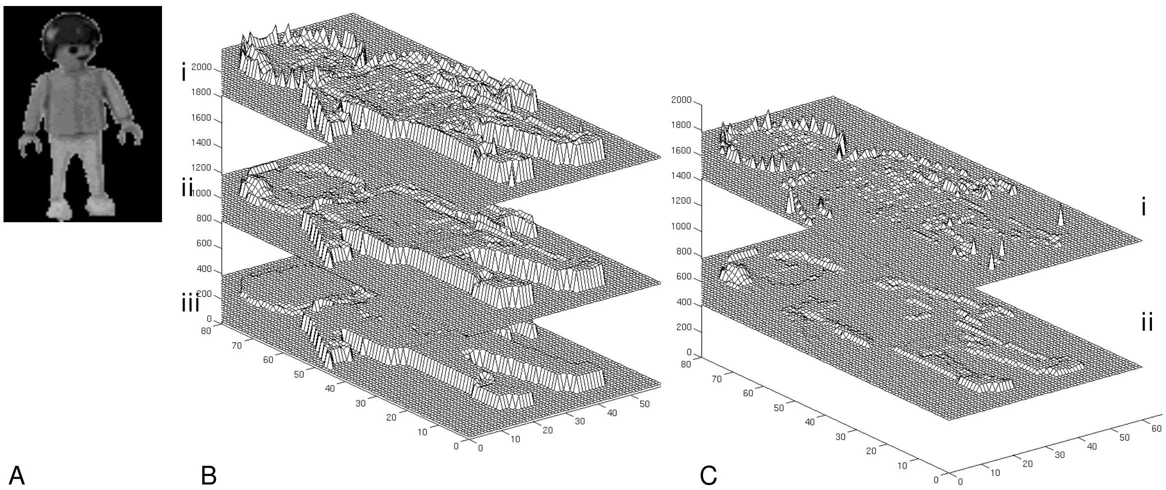

Figure 3: (A) Shows the original image I; (B)(i) shows, in relief, the original image, B(ii) shows $M^4(I)$ and B(iii) shows $M^{25}(I)$. (C) shows the differences between B(i), B(ii) and B(iii)

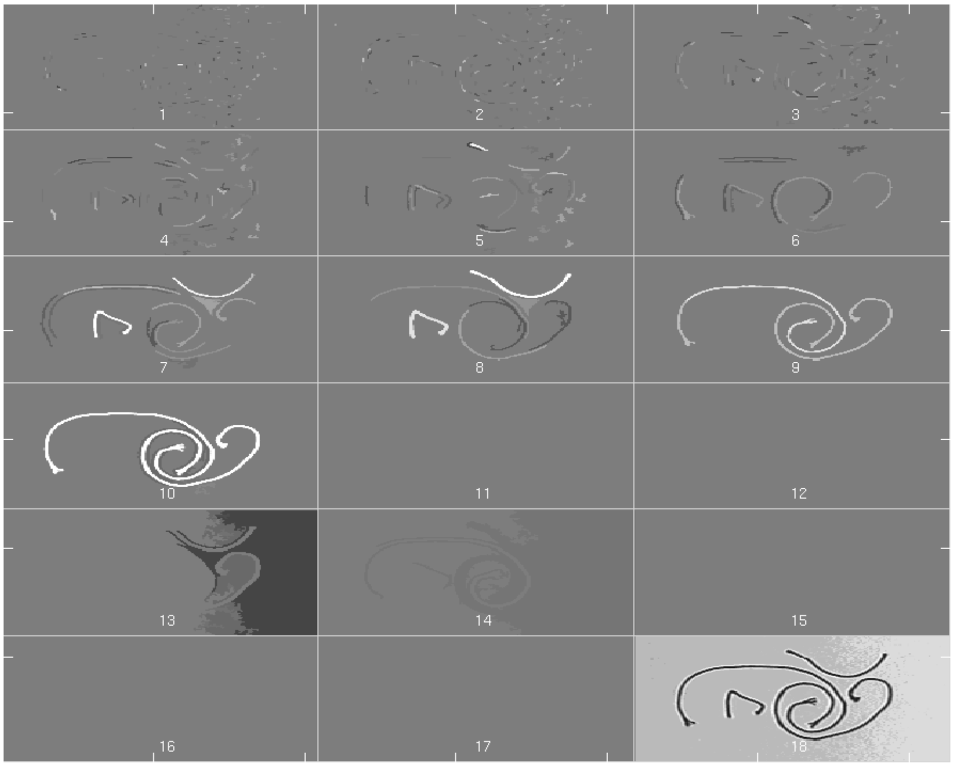

Figure 4: Panel 18 shows the original image and panels 1–17 show an area decomposition

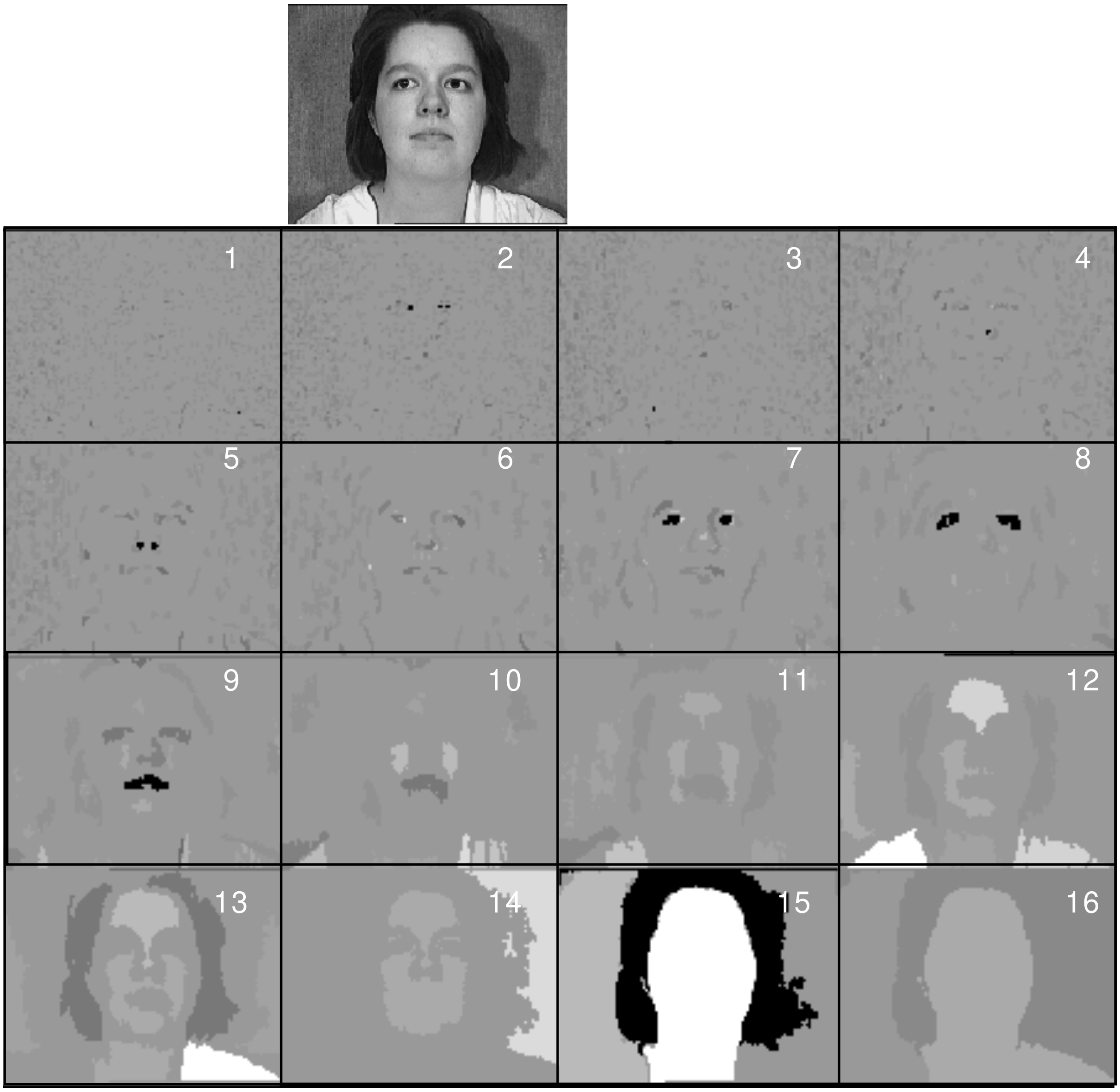

Figure 5: An area decomposition of an image (top) into a number of channels

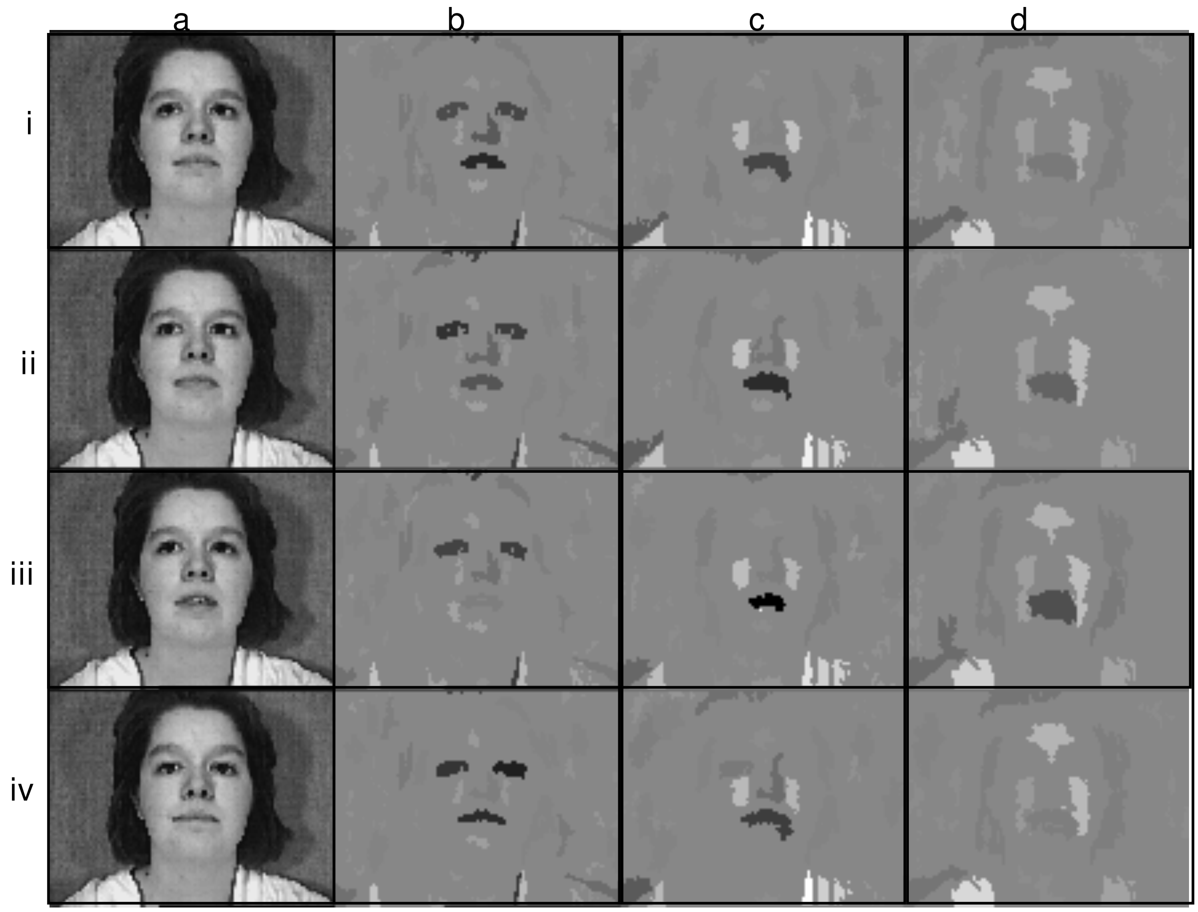

Figure 6: Frames 18,19,20, and 48 from a 25 frames/s sequence are in column a. Columns b,c and d show channels 9, 10 and 11. The mouth changes area.

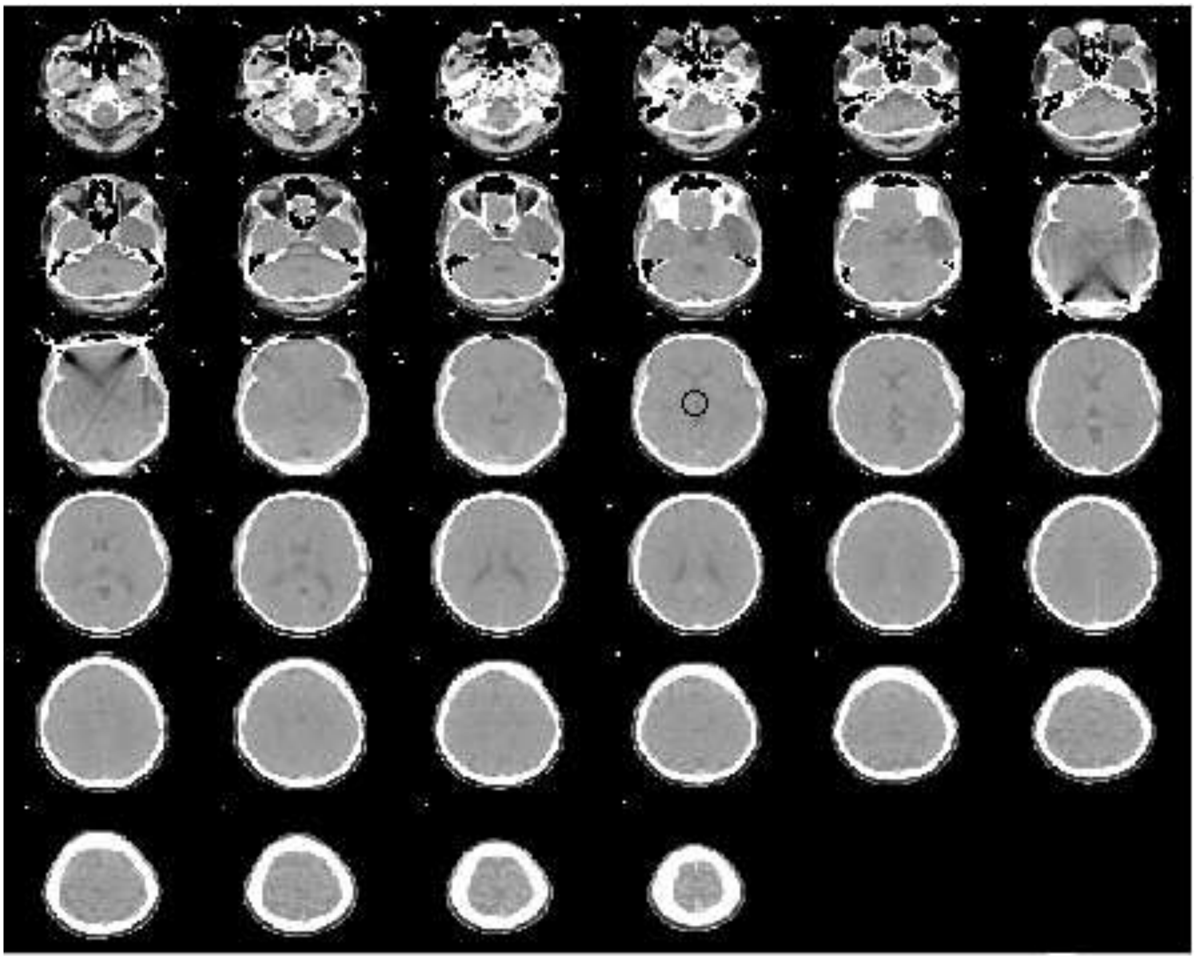

Figure 7: Thirty-five Computer tomography scans through a human head. The third ventricle is circled in the sixteenth slice

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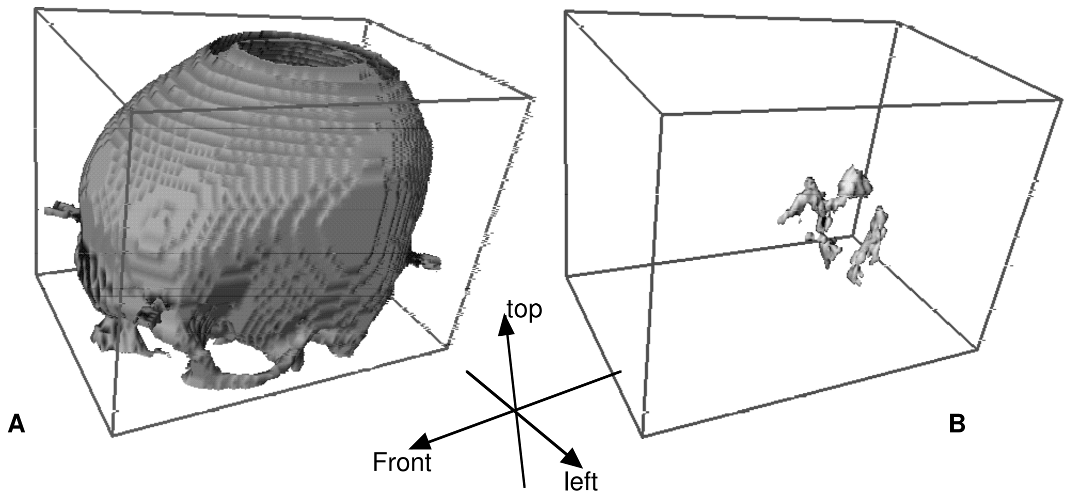

Figure 8: (A) shows a rendered 3D image of the skull. (B) shows the third ventricle on the same co-ordinated system as (A)