Minimal and Optimal Relative Normalization in Orthogonal Expression Reduction Systems

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Abstract

This paper continues an investigation into the theory of normalization relative to desirable sets $\mathcal{S}$ of 'normal forms', started in [GlKh94], where it is shown that such sets must be *stable*. In particular, we prove existence of *minimal* normalizing reductions for *regular stable* sets of normal forms, and give a sufficient and necessary criterion for a normalizing (w.r.t. a regular stable $\mathcal{S}$) reduction to be minimal. The sets of normal forms, head-normal-forms, weak head-normal-forms, etc. are all regular stable sets. As a corollary, we obtain the *Relative Standardization* theorem. Further, we generalize Lévy's *Optimality* theorem to the case of all stable sets of normal forms, and establish a relationship between relative minimal and optimal reductions, revealing a conflict between minimality and optimality of a reduction – for regular stable sets of normal forms, a term need not posses a reduction that is minimal and optimal at the same time. We also introduce and study a relative notion of *essentiality* and relate it to the relative notion of neededness. This study gives a clearer picture of the character of redexes that are contracted in minimal $\mathcal{S}$-normalizing reductions.

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1 Introduction

A normalizable term, in a rewriting system, may have an infinite reduction, so it is important to have a *normalizing* strategy which enables one to construct reductions to normal form. It is well known that the leftmost-outermost strategy is normalizing in the λ-calculus.

For Orthogonal Term Rewriting Systems (OTRSs), a general normalizing strategy, called the *needed* strategy, was found by Huet and Lévy in [HuLé91]. The needed strategy always contracts a *needed* redex – a redex whose residual is to be contracted in any reduction to normal form. Huet and Lévy showed that any term t not in normal form has a needed redex, and that repeated contraction of needed redexes in t leads to its normal form whenever there is one.

Barendregt et al. [BKKS87] generalized the concept of neededness to the λ-calculus. They studied neededness not only w.r.t. normal forms, but also w.r.t. head-normal forms – a redex is *head-needed* if its residuals are contracted in each reduction to a head-normal form. The authors proved correctness of the two needed strategies for computing normal forms and head-normal forms, respectively. In [Mar92], Maranget studied also a strategy that computes a weak head-normal form of a term in an OTRS. Normalization w.r.t. another interesting set of 'normal forms', that of constructor head-normal forms in constructor OTRSs, is studied by Nöcker [Nök94].

In [GlKh94], the present authors address the question of normalization with respect to a desired set of final terms, considering the properties that a set of terms must possess in order for the neededness theory of Huet and Lévy still to make sense. They introduce the notion of *neededness* w.r.t. a set of reductions Π or a set of terms $\mathcal{S}$ so that each existing notion of neededness can be given by specifying Π or $\mathcal{S}$. For example, *Huet&Lévy-neededness* is neededness w.r.t. the set of normal forms, *Maranget-neededness* is neededness w.r.t. all fair reductions, *head-neededness* is neededness w.r.t. the set of head-normal forms, etc. This work was performed in the context of Orthogonal *Expression Reduction Systems* (OERSs) [Kha90, Kha92], a form of higher-order rewriting which subsumes TRSs and the λ-calculus.

Natural conditions are imposed on $\mathcal{S}$, called *stability*, that are sufficient and necessary so that each $\mathcal{S}$-normalizable term not yet in $\mathcal{S}$ (not in $\mathcal{S}$-normal form) has at least one $\mathcal{S}$-needed redex, and so that repeated contraction of $\mathcal{S}$-needed redexes in a term t will lead to an $\mathcal{S}$-normal form of t whenever there is one. It is shown also that if a stable $\mathcal{S}$ is *regular*, i.e., if $\mathcal{S}$-unneeded redexes cannot duplicate $\mathcal{S}$-needed ones, then the $\mathcal{S}$-needed strategy is hypernormalizing as well.

The contribution of this paper is to develop a theory of *minimal* and *optimal* reduction in the framework of relative normalization, and to establish a relationship between them. While normal forms are unique in an OERS, a term may have many $\mathcal{S}$-normal forms. A reduction $P : t \twoheadrightarrow s$ with $t \notin \mathcal{S}$ and $s \in \mathcal{S}$ is said to be *$\mathcal{S}$-minimal* if it does no more work than any other $\mathcal{S}$-normalizing reduction $Q : t \twoheadrightarrow e$, i.e., the *residual* [Lév78] P/Q of P under Q is empty. The final term in the $\mathcal{S}$-minimal reduction is said to be an *$\mathcal{S}$-minimal* $\mathcal{S}$-normal form. $\mathcal{S}$-minimal $\mathcal{S}$-normal forms are useful to compute since any other $\mathcal{S}$-normal form is accessible from the $\mathcal{S}$-minimal one. Our research on minimal $\mathcal{S}$-normalizing reductions was inspired by a result of Maranget [Mar92], stating that *standard* reductions are minimal among reductions computing a 'stable prefix' of a given term. Clearly this concept of minimality is not useful for full normal forms.

We show that, for any stable and regular $\mathcal{S}$, any $\mathcal{S}$-normalizable term not yet in $\mathcal{S}$ possesses an $\mathcal{S}$-needed *$\mathcal{S}$-unabsorbed* redex, and repeated contraction of such redexes gives M-minimal $\mathcal{S}$-normalizing reductions. These redexes play the role of *standard* redexes w.r.t. $\mathcal{S}$, since they are $\mathcal{S}$-needed and cannot be duplicated. We further give a sufficient and necessary criterion for an

$\mathcal{S}$-normalizing reduction to be $\mathcal{S}$-minimal. We show also that $\mathcal{S}$-minimal reductions need not exist if $\mathcal{S}$ is stable but is irregular.

To give the characterization of $\mathcal{S}$-minimal reductions, we introduce a relative notion of *essentiality*. A subterm $s \subseteq t$ is said to be $\mathcal{S}$-essential if it has a *descendant* under at least one $\mathcal{S}$-normalizing reduction of t. The notion of descendant is a refinement of that of *residual* – the descendant of a contracted redex is its contractum, while it does not have residuals. The notion of essentiality (i.e., w.r.t. normal forms) was introduced in [Kha93, Kha88], where the essential strategy was shown to be normalizing for OTRSs and the λ-calculus (the strategy works for OERSs as well [GlKh94]).

For the case of normal forms, essentiality coincides with Maranget's concept of neededness, while in the general case, for stable $\mathcal{S}$, the two concepts are different. All $\mathcal{S}$-needed redexes are $\mathcal{S}$-essential. $\mathcal{S}$-unneeded redexes may be $\mathcal{S}$-essential, in which case they contribute to the minimal $\mathcal{S}$-normal form, or can be $\mathcal{S}$-inessential in which case they will be erased under any reduction to $\mathcal{S}$-normal form. Inessentiality is related to the notion of garbage – an inessential subterm may be replaced by any term without affecting the normal form. However, for certain stable and even regular $\mathcal{S}$, replacing an $\mathcal{S}$-inessential subterm by another term may lead to an $\mathcal{S}$-essential subterm at the same position, and the $\mathcal{S}$-normal form of the term may be changed.

We show that an $\mathcal{S}$-normalizing reduction is $\mathcal{S}$-minimal iff it contracts $\mathcal{S}$-*secured* redexes, i.e., the redexes that do not have residuals under any $\mathcal{S}$-normalizing reduction. Further, we show that a redex u in t is $\mathcal{S}$-secured iff every one of its residuals is either $\mathcal{S}$-needed or $\mathcal{S}$-inessential.

Our study of optimal normalization w.r.t. stable sets $\mathcal{S}$ is a generalization of Lévy's optimality theory [Lév80], developed for the λ-calculus. The generalization is rather straightforward, since a suitable labelling system allowing us to define a family relation in OERSs, due to Kennaway&Sleep [KeSl89], is available. We show that complete $\mathcal{S}$-needed family-reductions, which contract all members of a *family* containing an $\mathcal{S}$-needed redex in a multi-step, are *optimal* in the sense that they reach $\mathcal{S}$ in the least number of family-reduction steps.

It is easy to see that $\mathcal{S}$-needed complete family reductions, though optimal, need not be $\mathcal{S}$-minimal, because they may contract $\mathcal{S}$-unneeded redexes that are $\mathcal{S}$-essential. It is tempting to think that contracting only the $\mathcal{S}$-needed redexes of $\mathcal{S}$-needed families could yield $\mathcal{S}$-optimal reductions that are $\mathcal{S}$-minimal at the same time. We show however that this is not the case either in the λ-calculus or in OTRSs.

In section 2, we introduce higher order rewriting through *Expression Reduction Systems*. In section 3, we review the theory of relative normalization [GlKh94]. In section 4, we study $\mathcal{S}$-minimal reductions for regular stable sets $\mathcal{S}$. In section 5, we study the relative notion of essentiality, and in section 6, reductions standard w.r.t. regular stable sets. In section 7, we prove the Relative Optimality theorem. Finally, in section 8, we relate relative optimal and minimal reductions.

2 Orthogonal Expression Reduction Systems

Klop introduced *Combinatory Reduction Systems* (CRSs) in [Klo80] to provide a uniform framework for reductions with substitutions (also referred to as higher order rewriting) as in the λ-calculus [Bar84]. Several interesting formalisms have been introduced later [Nip93, OR94]. We refer to Klop et al. [KOR93] for a survey. Here we use a system of higher order rewriting, *Expression Reduction Systems* (ERSs), as defined in [GlKh94].