Higher Order Recursive Program Schemes are Turing Incomplete

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Abstract

We define *Higher Order Recursive Program Schemes* (HRPSs) by allowing metasubstitutions (as in the λ-calculus) in right-hand sides of function and quantifier definitions. To study reductions in a HRPS we split it into "first order part" and "pure substitution part". A study of pure substitutions and several kinds of *similarity* of redexes makes it possible to lift properties of (first order) Recursive Program Schemes to the higher order case. The crucial properties are that corresponding arguments of *essentially similar* redexes are either both essential or both inessential, and that essentially similar redexes create essentially similar redexes. The main result is the decidability of weak normalization in HRPSs, which immediately implies that HRPSs do not have full computational power. We analyze the structural properties of HRPSs and introduce several kinds of *persistent* higher order rewrite systems that enjoy similar properties. For *uniformly* persistent systems, we design an efficient optimal sequential normalizing strategy.

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1. Introduction

Higher Order Recursive Program Schemes (HRPSs) are recursive definitions of functions, predicates, and quantifiers, considered as rewriting systems. Similar definitions are used when one extends a theory by introducing new symbols. For example, the existential quantifier $\exists$ and the quantifier $\exists!$ for "there is exactly one" can be defined using Hilbert's operator (sign) τ as follows:

$$\exists aA \Leftrightarrow (\tau aA/a)A$$

$$\exists!aA \Leftrightarrow \exists aA \wedge \forall a\forall b(A \wedge (b/a)A \Rightarrow a = b)$$

The definitions of new symbols are added to the theory as axioms, and weak normalization of the system of definitions considered as a rewriting system ensures that the enriched theory is a conservative extension of the original one. A study of definitions as contraction schemes is made in [18].

The main result of the paper — decidability of weak normalization (that is, existence of a normal form) in HRPSs — implies that HRPSs do not have full computational power: the existence of an interpreter for, say λ-calculus, in an HRPS would imply decidability of weak normalization for λ-terms, which is not valid [3]. Note that rewrite rules for the conditional cannot be used for evaluating expression on the syntactic level, since they do not fall into the scope of our HRPSs.

Indeed, it is well-known that addition of $if - then - else$ yields full computational power (see for example Courcelle [4]). The deep reason for Turing incompleteness of HRPSs is the restricted possibility for redex creation compared to languages with full computational power, the λ-calculus and Combinatory Logic, for example. This paper is a part of a general study of how various kinds of redex creation are reflected in syntactic properties of rewriting systems such as normalization, perpetuality, decidability of weak and strong normalization (that is, termination of all reductions), expressive power, etc. Some results in this direction are obtained in [10].

In [9], we introduced a formalism for higher order rewriting (i.e., term rewriting systems with bound variables and substitution mechanism) which is close to Combinatory Reduction Systems (CRSs) of Klop [11]. Our syntax is more close to the syntax of λ-calculus and First Order Logic. For example, the β-rule is written as

$$\beta : Ap(\lambda aA, B) \to (B/a)A,$$

where a is to be instantiated by a variable and A and B are to be instantiated by terms. The expression $(B/a)A$ is a *metasubstitution*, and its instance $(t/x)s$ denotes the result of substitution of the term t for x in the term s. To express "pure" substitutions syntactically (instead of metasyntactically) we introduce S-reduction rules

$$S^{n+1}a_1 \dots a_nA_1 \dots A_nA_0 \to (A_1/a_1, \dots, A_n/a_n)A_0, \;\; n = 1, 2, \dots$$

The *substitution operator* S^{n+1} binds free variables only in the last argument. The difference with β-rules is that S-reductions can only express β-developments of λ-terms [8]. Therefore S-reductions have more simple properties and their study enables us to construct an interpretation of HRPS in (first order) Recursive Program Schemes (RPSs).

To this end, we split an HRPS R into a "first order part" R_f and a "substitution part" $\underline{S}$. (The method was originally used by Klop [11] to prove the Church-Rosser Property. Klop used β-reductions instead of $\underline{S}$-reductions.) The R_f-rules are obtained from R-rules by replacing in right-hand sides all metasubstitutions of the form $(A_1/a_1, \dots, A_n/a_n)A_0$ by $\underline{S}a_1 \dots a_nA_1 \dots A_nA_0$, respectively, and the $\underline{S}$-rules have the form

$$\underline{S}^{n+1}a_1 \dots a_nA_1 \dots A_nA_0 \to (A_1/a_1, \dots, A_n/a_n)A_0, \;\; n = 1, 2, \dots.$$

So $\underline{S}$-rules are just S-rules: we only need to distinguish S-redexes that are created during R_f-reduction steps. For example, a β-reduction step $(\lambda x\, t)s \to (s/x)t$ expands to a β_f-reduction step $(\lambda x\, t)s \to \underline{S}x\, s\, t$ and an $\underline{S}$-step $\underline{S}x\, s\, t \to (s/x)t$. Now in the promised interpretation of HRPSs in RPSs, $\underline{S}$-reductions play the role of projection functions. Ignoring some subtilities, the interpretation can be expressed by the following Representation Lemma: for any R-reduction $P : t \twoheadrightarrow s$ there is an R_f-reduction $P_f : t \twoheadrightarrow o$ such that s is obtained from o by any normalizing $\underline{S}$-reduction $P_{\underline{S}}$; and conversely (see Lemma 3.9 for the precise formulation). Moreover, P is *strictly equivalent* to the concatenation $P_f + P_{\underline{S}}$ of P_f and $P_{\underline{S}}$ in the following sense: *descendants* of any subterm of t under P and $P_f + P_{\underline{S}}$ are the same occurrences in s. The notion of descendant is a generalization of the usual notion of *residuals of redexes* to all subterms in a way that makes it possible to trace contracted redexes: the descendant of a contracted redex is its contractum, while it does not have residuals. For the case of TRSs, this notion is studied in [10].

As in the case of orthogonal Term Rewriting Systems (TRSs), we say that a subterm s of a term t in a CRS R is *inessential* if there is a reduction P starting from t such that s does not have descendants under P; we call s *essential* otherwise. Now an immediate corollary of the

Representation Lemma is that s is R-inessential in t iff there is an R_f-reduction $P : t \twoheadrightarrow e$ such that each P-descendant of s is $\underline{S}$-inessential in e. Since $\underline{S}$-reductions (like β-developments) are terminating, it is decidable whether a subterm is $\underline{S}$-inessential.

We can now introduce the key concepts of the paper: several kinds of *similarity* of redexes. We explain the notion informally in terms of β-redexes. Let us consider the redexes $u = (\lambda x.x)s$, $v = (\lambda x.(\lambda y.z)x)o$, $w = (\lambda x.xx)e$, and $u' = (\lambda x.y)t$. We call u and v *similar* (written $u \sim v$) because the binding variable x occurs free both in the bodies of u and v. In contrast, we have $u \not\sim u'$. We call u and w *essentially similar* (written $u \approx w$) because the binding variable x has essential occurrences in bodies of both redexes. Similarly, $v \approx u'$, because x occurs in the body $(\lambda y.z)x$ of v, but the occurrence is inessential — it does not have descendants in the contractum of $(\lambda y.z)x$. In general, let us call a β-redex $(\lambda x.e)o$ an *E-redex* if x has an essential occurrence in e. (Recall that $(\lambda x.e)o$ is an I-redex if x occurs free in e and is a K-redex otherwise.) Then I-redexes are similar, K-redexes are both similar and essentially similar, and E-redexes are essentially similar. An important property of E-redexes is that they stay E-redexes under any reduction. Further refinement of the notion of similarity takes into account also the number of occurrences and essential occurrences of a binding variable in corresponding arguments. These similarity notions are formalized by special *characteristic systems.* For example, the *essential characteristic system* of a redex $\sigma x_1 \ldots x_n t_1 \ldots t_m$ corresponding to a rewrite rule $\sigma a_1 \ldots a_n A_1 \ldots A_m \to B$ is the set of all pairs (a_i, A_j) such that t_j is in the scope of σ and x_i has an essential occurrence in t_j. We call a pair $(r, ECS(r))$, where r is a rule and $ECS(r)$ is an essential characteristic system for some r-redex, an *essentially characterized rule* (*EC-rule* for short).

Let a term t be obtained from s by replacing some subterms. We call t and s *essentially similar* if corresponding redexes in t and s (that are outside of replaced subterms) are essentially similar. The crucial property of essentially similar terms is the following Essential Similarity Lemma: corresponding subterms of essentially similar terms are either both essential or both inessential. A consequence is that corresponding arguments of essentially similar redexes are either both essential or both inessential. Hence each EC-rule $(r, ECS(r))$ has an *essentiality indicator* which indicates which arguments of a redex with the essential characteristic system $ECS(r)$ (*$(r, ECS(r))$-redex* for short) are essential. Another consequence of the Essential Similarity Lemma is that essentially similar redexes create essentially similar redexes.

Now, the strategy for proving the decidability of weak normalization in HRPSs is similar to the first order case [10]. We first establish a criterion for weak normalization. We define an *essential chain* to be a sequence of EC-rules $(r_0, ECS(r_0)), (r_1, ECS(r_1)), \ldots$ such that any $(r_i, ECS(r_i))$-redex creates an $(r_{i+1}, ECS(r_{i+1}))$-redex $(i = 0, 1, \ldots)$. We show that a term in an HRPS R is weakly normalizable iff any essential chain, starting from any EC-rule $(r, ECS(r))$ such that a $(r, ECS(r))$-redex has an essential occurrence in t, is finite. So we need an algorithm for finding all essential (and inessential) subterms in a given term, which also will enable us to find essential characteristic systems of all redexes in t, and an algorithm for finding essentiality indicators of all EC-rules. Complete descriptions of the algorithms are presented in the body of the paper; we only note that our algorithm can also find essentiality indicators of redexes that do not have normal forms. As a corollary, we have an algorithm for optimal S-reductions and, in general, algorithms for optimal developments in orthogonal CRSs.

The above properties of HRPSs are due to the fact that each subterm s of a term t is *free*: if s is inside a redex u, then it is either inside an argument of u or coincides with u. We introduce *persistent* CRSs (PCRSs), where not all subterms are free but all free subterms remain free under any reduction. This persistency property of free subterms is due to the fact that PCRSs are the systems where only a special kind of redex creation — *generation* — is possible: all new redexes are actually presented in right-hand sides of rewrite rules. For example, in the step $\sigma xx \rightarrow f((\sigma xx/x)x) = f(\sigma xx)$, which corresponds to a rule $\sigma aA \rightarrow f((\sigma aA/a)A)$ and an assignment $\theta(a) = \theta(A) = x$, the created redex σxx is an instance of the metaterm σaA presented in the right-hand side of the rewrite rule. We have generation here because $\theta(a)$ has an occurrence in $\theta(A)$. Therefore, the generation is not *uniform* — uniformly generated redexes are presented in right-hand sides of rules outside of "mobile" arguments of metasubstitutions. During the reduction step $\sigma xx \rightarrow f((d/x)x) = f(d)$ in the system $\{r_1 : \sigma aA \rightarrow f((d/a)A),\ r_2 : f(d) \rightarrow c\}$, where d and c are constants, we have *quasi-generation* in the sense that all symbols needed to create the new redex $f(d)$ are presented in the right-hand side of r_1.

The Essential Similarity Lemma remains valid in PCRSs if the replaced subterms are free. Therefore, all the results obtained for HRPSs, the decidability of weak normalization in particular, remain valid for PCRSs in general. Similar reasoning shows that strong normalization also is decidable in PCRSs. For uniformly persistent systems, we show that the strategy that in each step contracts an innermost essential redex is optimal. For PCRSs in general, sharing of redexes of the same origin [14] is necessary, and this can be implemented in the framework of Interaction Systems [2]. However, the above results can not be generalized to *quasi-persistent* CRSs, where quasi-generation of redexes is possible.

2. Orthogonal Combinatory Reduction Systems

Combinatory Reduction Systems have been introduced in Klop [11] to provide a uniform framework for reductions with substitutions, as in the λ-calculus and its extensions [3]. Different formalisms are proposed in Kennaway and Sleep [7] ((Functional) Combinatory Reduction Systems), Khasidashvili [8] (Expression Reduction Systems), and Nipkow [16] (Higher-order Rewrite Systems). They are extensions of Term Rewriting Systems [5, 12] by means of variable binding and substitution mechanisms. Restricted notions of CRSs were first introduced in Pkhakadze [18] and Aczel [1]. A comparison of some formalisms of rewriting systems with bound variables and substitution mechanism (referred to also as higher order rewrite systems) can be found in van Oostrom and van Raamsdonk [17]. A survey paper is Klop et al. [13]. Here we describe a system for higher order rewriting as defined in Khasidashvili [9]; it is based on the syntaxs of [18].

Definition 2.1 (1) Let Σ be an *alphabet*, comprising *variables* $v_0, v_1, \ldots$; *function symbols*, also called *simple operators*; and *operator signs* or *quantifier signs*. Each function symbol has an *arity* $k \in N$, and each operator sign σ has an *arity* (m, n) with $m, n \neq 0$ such that, for any sequence $x_1, \ldots, x_m$ of pairwise distinct variables, $\sigma x_1 \ldots x_m$ is a *compound operator* or a *quantifier* with *arity* n. Occurrences of $x_1, \ldots, x_m$ in $\sigma x_1 \ldots x_m$ are called *binding variables*. Each quantifier $\sigma x_1 \ldots x_m$, as well as corresponding quantifier sign σ and binding variables $x_1 \ldots x_m$, has a *scope indicator* $(k_1, \ldots, k_l)$ to specify the arguments in which $\sigma x_1 \ldots x_m$ binds all free occurrences of $x_1, \ldots, x_m$. *Terms* are constructed from variables using functions and quantifiers in the usual way.

(2) *Metaterms* are constructed from *terms*, *term metavariables*, which range over terms, and *object metavariables*, which range over variables. Apart from the usual rules for term-formation, one is allowed to have *metasubstitutions* — expressions of the form $(A_1/a_1, \ldots, A_n/a_n)A_0$, where a_i are object metavariables and A_j are metaterms. Metaterms that do not contain metasubstitutions are called *simple metaterms*. An *assignment* maps each object metavariable to a variable and each term metavariable to a term over Σ. If t is a metaterm and θ is an assignment, then the *θ-instance* $t\theta$ of t is the term obtained from t by replacing metavariables with their values under θ, and by replacement of subterms of the form $(t_1/x_1, \ldots, t_n/x_n)t$ by the result of substitution of

terms $t_1, \dots, t_n$ for free occurrences of $x_1, \dots, x_n$ in t.

Definition 2.2 (1) A *Combinatory Reduction System* (CRS) is a pair (Σ, R), where Σ is an alphabet, described in Definition 2.1, and R is a set of rewrite rules $r : t \to s$, where t and s are metaterms such that t is a simple metaterm and is not a metavariable, and each term metavariable which occurs in s occurs also in t. Further,

(a) The metaterms t and s do not contain variables, and each occurrence of object metavariable in t and s is bound.

(b) An occurrence of a term-metavariable A in s is in the scope of an occurrence of an object metavariable a in s iff any occurrence of A in t is in the scope of an occurrence of a in t.

(2) An assignment θ is *admissible* for a rule $r : t \to s$ if occurrences of binding variables in $s\theta$ corresponding to object metavariables of s not occurring in t do not bind variables in subterms corresponding to term metavariables of s. For any admissible assignment θ', $t\theta'$ is an *r-redex*, and $s\theta$ is the *contractum* of $t\theta$. Redexes that are instances of the left-hand side of the same rule are called *weakly similar*.

(3) R is *simple* if right-hand sides of R-rules are simple.

Remark 2.1 Terms o and e are called *congruent*, notation $o \cong e$, if o is obtained from e by renaming bound variables. The conditions in Definition 2.2 imply that, for any rule $r : t \to s$, if $\theta, \theta' \in AA(r)$, then $t\theta \cong t\theta'$ implies $s\theta \cong s\theta'$. Below we identify all congruent terms.

Notation We use a, b for object metavariables, A, B for term metavariables, c, d for constants, t, s, e, o for terms and metaterms, u, v, w for redexes, σ for operators and operator signs, and P, Q for reductions. We write $s \subseteq t$ if s is a subterm of t. A one-step reduction in which a redex u in a term t is contracted is written as $t \xrightarrow{u} s$ or $t \to s$. We write $P : t \twoheadrightarrow s$ if P denotes a reduction of t to s. $|P|$ denotes the length, i.e., the number of steps, of P. If the last term of P coincides with the initial term of Q, then $P + Q$ denotes the concatenation of P and Q. $\emptyset_t$, or simply $\emptyset$, denotes the empty reduction of a term t; the symbol $\emptyset$ is also used to denote the empty set.

Definition 2.3 A term t in a CRS R is said to be in *normal form (nf)* or to be a nf if it does not contain redexes. If $s \twoheadrightarrow t$ and t is a nf, then t is called a normal form of s. A term is called *weakly normalizable* if it has a nf and is called *strongly normalizable* if it does not possess an infinite reduction. A CRS R is called *weakly normalizing* (resp. *strongly normalizing*) if each term in R is weakly normalizable (resp. strongly normalizable).

Definition 2.4 Let $t \to s$ be a rule in a CRS R and θ be an assignment. Subterms of a redex $v = t\theta$ that correspond to term metavariables of t are the *arguments* of v, and the rest is the *pattern* of v. Subterms of v rooted at the pattern are called the *pattern-subterms* of v. If R is a simple CRS, then arguments, pattern, and pattern-subterms are defined analogously in the contractum $s\theta$ of v.

Definition 2.5 A rewrite rule $t \to s$ in a CRS R is *left-linear* if t is *linear*, i.e., no term metavariable occurs more than once in t. R is *left-linear* if each rule in R is so. R is *non-ambiguous* or *non-overlapping* if in no term redex-patterns can overlap. R is *orthogonal* (OCRS) if it is left-linear and non-overlapping.

Definition 2.6 The CRS S comprises rules of the form

$$S^{n+1} a_1 \dots a_n A_1 \dots A_n A \to (A_1/a_1, \dots, A_n/a_n)A, \ n = 1, 2, \dots,$$

where S^{n+1} is the *operator sign of substitution* with arity $(n, n+1)$ and scope indicator $(n+1)$, and $a_1, \dots, a_n$ and $A_1, \dots, A_n, A$ are pairwise distinct object and term metavariables, respectively.

Definition 2.7 Let $R = \{r_i : t_i \to s_i | i \in I\}$ be a CRS.

1. $R_f =_{def} \{r'_i : t_i \to s'_i | i \in I\}$, where s'_i is obtained from s_i by replacing all metasubstitutions of the form $(t_1/a_1, \ldots, t_n/a_n)t$ with $\underline{S}^{n+1}a_1 \ldots a_n t_1 \ldots t_n t$, respectively ($\underline{S}^{n+1}$ is a fresh symbol with the same arity and scope indicator as S^{n+1}).

2. If R is simple, then $R_{fS} =_{def} R_f =_{def} R$. Otherwise $R_{fS} =_{def} R_f \cup \underline{S}$, where $\underline{S}$-rules are obtained from S-rules by underlining the S-symbols.

3. For each step $e = C[t_i\theta] \xrightarrow{u} C[s_i\theta] = o$ in R (corresponding to the rule r_i and an admissible assignment θ) there is a reduction $P : e = C[t_i\theta] \to C[s'_i\theta] \twoheadrightarrow C[s\theta] = o$ in R_{fS}, where $C[s'\theta] \twoheadrightarrow C[s\theta]$ is the rightmost innermost normalizing $\underline{S}$-reduction. We call P the *expansion* of u and denote it by $Ex(u)$. The notion of *expansion* generalizes naturally to R-reductions with 0 or more steps.

Definition 2.8 1. Let $t \xrightarrow{u} s$ in a simple OCRS and e be the contractum of u in s. For each argument t^* of u there are 0 or more arguments of e. We call them $(u-)$*descendants* of t^*. Correspondingly, subterms of t^* have 0 or more *descendants*. The *descendant* of each pattern-subterm of u that is not a variable is e. (We do not define descendants of "variable pattern-subterms", which are binding variables). It is clear what is to be meant under *descendants* of subterms that are not in u. The notion of *descendant* extends naturally to arbitrary reductions in simple OCRSs.

2. Let $t \xrightarrow{u} s$, where $u = Sx_1 \ldots x_n t_1 \ldots t_n t_0$, and let e be the contractum of u in s. For each mobile argument t_i of u ($i = 1, \ldots, n$) there are substituted occurrences of t_i in e. We call them *u-descendants* of t_i. By definition, they also are *u-descendants* of corresponding free occurrences of the variable x_i in t_0. Subterms in t_i have the same number (possibly none) of *descendants* in s. The *descendant* of u is e. It is clear what is to be meant under *descendants* of subterms that are not in u, or are in t_0 and are not free occurrences of variables $x_1, \ldots, x_n$. The notion of *descendant* extends naturally to S-reductions with 0 or more steps.

3. Let $P : t \twoheadrightarrow s$ in an OCRS R and let $Q = Ex(P)$. It is clear from (1) and (2) what is to be meant under Q-descendants of subterms in t. We call a subterm $o' \in s$ a P-*descendant* of a subterm $o \in t$ if o' is a Q-descendant of o, and call o in this case a P-*ancestor* of o'.

4. Let $t \xrightarrow{u} s$. Descendants of all redexes of t except u are also called *residuals*. By definition, u does not have *residuals* in s. A redex $v \subseteq s$ is a (u)-*new* redex or a *created* redex if it is not a residual of a redex in t. The notion of *residual* of redexes extends naturally to reductions with 0 or more steps.

Definition 2.9 We call the co-initial reductions $P : t \twoheadrightarrow s$ and $Q : t \twoheadrightarrow e$ *strictly equivalent* (written $P \approx_{st} Q$) if $s = e$ and P-descendants and Q-descendants of any subterm of t are the same in s and e.

Notation If F is a set of redexes in t and $P : t \twoheadrightarrow s$, then F/P denotes the set of all residuals of redexes from F in s. If $F = \{u\}$, then we write u/P for $\{u\}/P$. In the following, F will also denote a complete F-development, where the residuals of redexes from F are contracted as long as possible. Similarly, if $u \in t$, then u will also denote the reduction $t \xrightarrow{u} s$.

Definition 2.10 Let $Q : t \twoheadrightarrow s$ and $t \xrightarrow{u} e$. Then the residual Q/u of Q by u is defined modulo permutation of non-overlapping steps by induction on $|Q|$ as follows. If $Q = \emptyset_t$, then $Q/u = \emptyset_e$. If $Q = Q' + v$, then $Q/u = Q'/u + v/(u/Q')$.

Definition 2.11 Let $P : t \twoheadrightarrow s$ and $Q : t \twoheadrightarrow e$. Then the residual P/Q of P by Q and the residual Q/P of Q by P are defined modulo permutation of non-overlapping steps by induction on $|P|$ as follows.

(1) If $P = \emptyset_t$, then $P/Q = \emptyset_e$ and $Q/P = Q$.

(2) If $P = P' + u$, then $P/Q = P'/Q + u/(Q/P')$ and $Q/P = (Q/P')/u$.

We write $P \sqcup Q$ for $P + Q/P$.

Theorem 2.1 (**Strict Church-Rosser** [9]) Let R be an OCRS, and P and Q be co-initial reductions in R. Then $P \sqcup Q \approx_{st} Q \sqcup P$.

3. Similarity of redexes

In this section, we study some properties of substitutions and similarity of terms. In particular we prove the Replacement Lemma and the Essential Similarity Lemma.

Definition 3.1 We call an OCRS R a *Higher Order Recursive Program Scheme* (HRPSs) if, for any rule $t \rightarrow s$, pattern of t consists of one operator, i.e., t has the form $\sigma a_1 \dots a_m A_1 \dots A_n$, where σ is an operator sign with arity (m, n) (σ is a function if $m, n = 0$).

Definition 3.2 Let t be a term in an OCRS R. We call a subterm s in t *essential* (written $ES(s,t)$) if s has at least one descendant under any reduction starting from t and call it *inessential* (written $IE(s,t)$) otherwise.

The notion of essentiality is a generalization of the notion of *neededness* [6, 15] in a way that it works for all subterms, bound variables in particular. (See [10] for the precise relationship between the notions). The following two lemmas are valid for all OCRSs; the proofs are similar to the case of orthogonal TRSs [10].

Lemma 3.1 Let $s_0, \dots, s_k \subseteq t$ be such that $IE(s_i, t)$ for all $i = 0, \dots, k$. Then there exists a reduction P starting from t such that none of the subterms $s_0, \dots, s_k$ have P-descendants.

Proof Let P_i be a reduction starting from t such that s_i does not have P_i-descendants (P_i exists since $IE(s_i, t)$). Then, by Theorem 2.1, one can take $P = (\dots(P_1 \sqcup P_2) \sqcup \dots \sqcup P_n)$.

Lemma 3.2 Let $P : t \twoheadrightarrow t'$ and $s \subseteq t$. Then $IE(s, t)$ iff any P-descendant s' of s is inessential in t'. In particular, if t' is a normal form, then $ES(s,t)$ iff s has a P-descendant.

Proof ($\Rightarrow$) Let $IE(s,t)$. Then there is some reduction Q starting from t such that s does not have Q-descendants. By Theorem 2.1, $P + Q/P \approx_{st} Q + P/Q$. Hence, s' does not have P/Q-descendants, i.e., $IE(s', t')$. ($\Leftarrow$) If all u-descendants of s are inessential in t', then, by Lemma 3.1, there is some reduction P' starting from t' under which none of them have descendants. Thus s does not have $P + P'$-descendants, i.e., $IE(s,t)$.

Definition 3.3 We call a CRS R' a *subsystem* of a CRS R if the alphabet of R' is a subset of the alphabet of R and the set of R'-rules is a subset of the set of R-rules.

Notation Below $EFV_R(t)$ denotes the set of variables having R-essential free occurrences in t and $FV(s)$ denotes the set of variables having free occurrences in s. We write $t = (t_1//e_1, \dots, t_k//e_k)e$ if t is obtained from e by replacing non-overlapping proper subterms $e_1, \dots, e_n$ in e with $t_1, \dots, t_n$, respectively. For any $s \subseteq t$, $BV_R(s)$ (resp. $EBV_R(s)$) denotes the set of free occurrences (resp. R-essential free occurrences) of s bound by quantifiers belonging to patterns of R-redexes that are outside s. For any subsystems R_1 and R_2 of R, $EBV_{R_1,R_2}(s)$ is the set of R_2-essential free occurrences of s that are bound by quantifiers belonging to patterns of R_1-redexes that are outside s. (Note that $EBV_{R,\emptyset}(s) = BV_R(s)$ and $EBV_{R,R}(s) = EBV_R(s)$.)

Definition 3.4 (1) Let $\sigma a_1 \dots a_n A_1 \dots A_m$ be the left-hand side of a rewrite rule r in an HRPS R and let $\sigma x_1 \dots x_n t_1 \dots t_m$ be an r-redex. The *characteristic system* of u (written $CS(u)$) is the set of pairs (a_j, A_i) such that t_i is in the scope of σ and $x_j \in FV(t_i)$ ($i = 1, \dots, m$, $j = 1, \dots, n$). In this case, u is an $(r, CS(u))$*-redex*. A *characterized rule* (*C-rule* for short) is a pair $(r, CS(r))$, where $CS(r)$ is a characteristic system for some R-redex. For any subsystem R' of R, an R'*-essential characteristic system* of u (written $ECS_{R'}(u)$) is the set of pairs (a_j, A_i) such that t_i is in the scope of σ and $x_j \in EFV_{R'}(t_i)$. In this case, u is an $(r, ECS_{R'}(u))$*-redex*. An R'*-essentially characterized rule* (*EC-rule* for short) is a pair $(r, ECS_{R'}(r))$, where $ECS_{R'}(r)$ is an R'-essential characteristic system for some r-redex. (We omit the subscript R' if $R = R'$.)

(2) Let $t = (t_1//e_1, \dots, t_k//e_k)e$. We write $e \prec_R t$ (resp. $e \lll_R t$) if $BV_R(e_i) \subseteq BV_R(t_i)$ (resp. $EBV_R(e_i) \subseteq EBV_R(t_i)$) for all $i = 1, \dots, k$. We call the terms t and e R*-similar* (resp. R*-essentially similar*), written $e \sim_R t$ (resp. $e \approx_R t$), if $s \prec_R t$ and $t \prec_R s$ (resp. $s \lll_R t$ and $t \lll_R s$). We write $e \prec_{R_1,R_2} t$ if $EBV_{R_1,R_2}(e_i) \subseteq EBV_{R_1,R_2}(t_i)$ for all $i = 1, \dots, k$, and write $e \sim_{R_1,R_2} t$ if $e \prec_{R_1,R_2} t$ and $t \prec_{R_1,R_2} e$. In the latter case, we call e and t (R_1, R_2)*-essentially similar*. (Note that $\prec_R$ coincides with $\prec_{R,\emptyset}$ and $\lll_R$ coincides with $\prec_{R,R}$; therefore, R-similarity coincides with $(R, \emptyset)$-similarity and R-essential similarity coincides with (R, R)-similarity.)

Of course, (essential) similarity of terms depends on the specification of the replaced subterms. Note that, for any weakly similar R-redexes v and u, if one specifies replaced subterms to be the arguments of the redexes, then $v \prec u$ (resp. $v \lll u$) iff $CS(v) \subseteq CS(u)$ (resp. $ECS_R(v) \subseteq ECS_R(u)$). Below, when we speak of (essential) similarity of redexes without specifying replaced subterms, we mean that the replaced subterms are the arguments of the redexes; we write $v \overset{r}{\prec} u$, $v \overset{r}{\lll} u$, $v \overset{r}{\sim} u$, and $v \overset{r}{\approx} u$ for $v \prec u$, $v \lll u$, $v \sim u$, and $v \approx u$, respectively, to stress the fact that the replaced subterms are specified to be arguments of the redexes.

3.1 The Replacement Lemma

Definition 3.5 Let $u = Sx_1 \dots x_n t_1 \dots t_n t_0$ and t_0' be an S-normal form of t_0. A subterm e in u is called u*-inessential* (written $IE(u; e)$) if e is in t_i for some $1 \le i \le n$ and $x_i \notin FV(t_0')$.

Lemma 3.3 Let $u = Sx_1 \dots x_n t_1 \dots t_n t_0 \subseteq t$. Then $IE_S(u; t_i)$ iff $x_i \notin EFV_S(t_0)$.
Proof By Lemma 3.2, if t_0' is the S-normal form of t, then $EFV_S(t_0) = FV(t_0')$.

Lemma 3.4 Let $s \subseteq t$. Then $IE_S(s, t)$ iff $IE_S(u; s)$ for some S-redex u in t.
Proof sketch One can take for u the redex whose residual erases all descendants of s in the rightmost innermost normalizing S-reduction.

Lemma 3.5 Let $e \subseteq s \subseteq t$. Then $ES_S(e, t)$ iff $ES_S(e, s)$ and $ES_S(s, t)$.
Proof ($\Rightarrow$) From Definition 3.2. ($\Leftarrow$) By Lemma 3.4, the redex that would make e inessential can neither occur in s nor contain s in its argument.

Lemma 3.6 (**Replacement Lemma**) Let $s = (s_1//t_1, \dots, s_n//t_n)t$, where s_i and t_i do not contain S-redexes, and let $t \prec_S s$. Further, let s' and t' be any corresponding subterms in s and t that are not in replaced subterms. Then $IE_S(s', s) \Rightarrow IE_S(t', t)$.
Proof By induction on the length of s. If t and s are not S-redexes, then the lemma follows easily from Lemma 3.4 and the induction assumption. So suppose that $t = Sx_1 \dots x_m e_1 \dots e_m e_0$, $s = Sx_1 \dots x_m o_1 \dots o_m o_0$, $s' \subseteq o_l$, and $IE_S(s', s)$. If $IE_S(s', o_l)$, then by the induction assumption $IE_S(t', e_l)$ and hence $IE_S(t', t)$. Otherwise, by Lemma 3.5, we have $IE_S(o_l, s)$. Hence, by Lemma 3.4, $IE_S(s; o_l)$. Thus, by Lemma 3.3, $x_l \notin EFV_S(o_0)$. Let us show that $x_l \notin EFV_S(e_0)$.

By the induction assumption, if x_l has an S-essential occurrence in e_0 outside of replaced subterms, then the corresponding occurrence of x_l in o_0 is S-essential. If x_l has an S-essential occurrence in a subterm $t_j \subseteq e_0$, then, by Lemma 3.5, $ES_S(t_j, e_0)$. By the induction assumption, $ES_S(s_j, o_0)$. Since $t_j \prec_S s_j$, x_l has a free occurrence in s_j. Since s_j does not contain S-symbols, it follows from Lemma 3.4 that this occurrence is S-essential in s_j and hence, by Lemma 3.5 and $ES_S(s_j, o_0)$, is S-essential in o_0. Hence $x_l \notin EFV_S(e_0)$ and, by Lemma 3.3, $IE_S(t; e_l)$. Therefore, by Definition 3.5 and Lemma 3.4, $IE_S(t; t')$ and $IE_S(t', t)$.

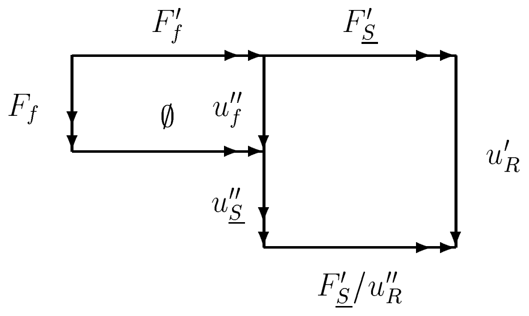
3.2 The Essential Similarity Lemma
In this subsection we only consider HRPSs.

Definition 3.6 Let $t = (t_1//s_1, \ldots, t_k//s_n)s$ and let $P : s = e_0 \xrightarrow{v_0} e_1 \xrightarrow{v_1} \ldots$ in R_f. Let us construct the reduction $P\|(t) : t = o_0 \xrightarrow{u_0} o_1 \xrightarrow{u_1} \ldots$ in R_f as follows. If v_0 is outside $s_1, \ldots, s_n$, then u_0 is its corresponding redex in o_0. Otherwise $u_0 = \emptyset$. Since o_1 is obtained from e_1 by replacing the descendants of $s_1, \ldots, s_n$ with $t_1, \ldots, t_n$, respectively, in o_1 we can choose the redex u_1 analogously, and so on.

Lemma 3.7 Let $t = (t_1//s_1, \ldots, t_n//s_n)s$, let $P : s \twoheadrightarrow s'$ in R_f, and let $Q = P\|(t) : t \twoheadrightarrow t'$. Then t' can be obtained from s' by replacing the descendants of $s_1, \ldots s_n$ with $t_1, \ldots, t_n$, respectively, and if e and o are corresponding subterms in s and t, then the P-descendants and Q-descendants of e and o are the corresponding subterms of s' and t'.
Proof By induction on $|P|$.

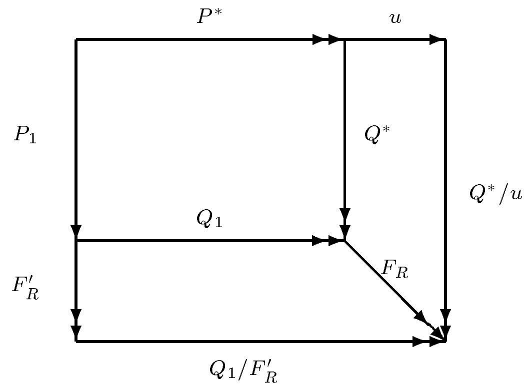
Notation Recall that, if F is a set of R-redexes in a term t, then a *complete F-development of t* is a reduction that contracts residuals of redexes from F as long as possible. Below F_R denotes a complete F-development in R, F_f denotes a complete F-development in R_f, and $F_{\underline{S}}$ denotes a complete development of the set of $\underline{S}$-redexes created during F_f. If F consists of one redex u only, then we write u_R, u_f, and $u_{\underline{S}}$ for F_R, F_f, and $F_{\underline{S}}$, respectively.

Lemma 3.8 Let F be a set of R-redexes in a term t. Then $F_R \approx_{st} F_f + F_{\underline{S}}$.
Proof By induction on number n of redexes in F. The case $n = 0$ is trivial. So let $F = F' \cup \{u\}$, where u is an outermost among redexes in F. Then $F_R \approx_{st} F'_R + u'_R$ (where $u' = u/F_R$) $\approx_{st}$(by the induction assumption)$\approx_{st} F'_f + F'_{\underline{S}} + u'_R \approx_{st} F'_f + u''_R + F'_{\underline{S}}/u''_R$ (where u'' is the unique ancestor of u') $\approx_{st} F'_f + u''_f + u''_{\underline{S}} + F'_{\underline{S}}/u''_R \approx_{st} F_f + u''_{\underline{S}} + F'_{\underline{S}}/u''_R \approx_{st} F_f + F_{\underline{S}}$.

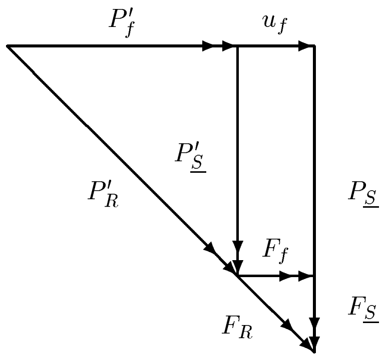


Lemma 3.9 (**Representation Lemma**) Let R be an HRPS.
(1) For any R-reduction $P : t \twoheadrightarrow_R s$ there are $Q : s \twoheadrightarrow_R e$ and $P' : t \twoheadrightarrow_{R_f} o$ such that $P + Q \approx_{st} P' + P''$, where P'' is a normalizing $\underline{S}$-reduction starting from o.
(2) If $P_f : t \twoheadrightarrow_{R_f} s$ and $P_{\underline{S}}$ is a normalizing $\underline{S}$-reduction starting from s, then there is an R-reduction P_R such that $P_f + P_{\underline{S}} \approx_{st} P_R$.

Proof (1) By induction on $|P|$. Let $P = P^* + u$. By the induction assumption, there are R-reduction Q^*, R_f-reduction P_1, and a normalizing $\underline{S}$-reduction Q_1 such that $P^* + Q^* \approx_{st} P_1 + Q_1$. Let F be the set of all Q^*-residuals of u and let F' be the set of their Q_1-ancestors ($\underline{S}$-reduction steps do not create new redexes). Then, $F'_R + Q_1/F'_R \approx_{st} Q_1 + F_R$. By Lemma 3.8, $F'_R = F'_f + F'_{\underline{S}}$. Thus $P_1 + F'_f + F'_{\underline{S}} + Q_1/F'_R \approx_{st} P_1 + Q_1 + F_R \approx_{st} P^* + Q^* + F_R \approx_{st} P^* + u + Q^*/u = P + Q^*/u$ and we can take $Q = Q^*/u$, $P' = P_1 + F'_f$, and $P'' = F'_{\underline{S}} + Q_1/F'_R$.



(2) By induction on $|P_f|$. The case $|P_f| = 1$ is immediate. So let $P_f : t \twoheadrightarrow e \xrightarrow{u} s$ and let $P'_f : t \twoheadrightarrow e$ be the initial part of P_f. By the induction assumption, there is an R-reduction P'_R such that $P'_R \approx_{st} P'_f + P'_{\underline{S}}$. Hence $P_f + P_{\underline{S}} \approx_{st} P'_f + u_f + P_{\underline{S}} \approx_{st} P'_f + P'_{\underline{S}} + F_f + F_{\underline{S}}$ (where F is the set of residuals of u under $P'_{\underline{S}}$) $\approx_{st}$ (by the induction assumption) $\approx_{st} P'_R + F_f + F_{\underline{S}} \approx_{st}$ (by Lemma 3.8)$\approx_{st} P'_R + F_R$, and we can take $P_R = P'_R + F_R$.



Remark 3.1 It is easy to see that the reduction Q in the Representation Lemma contracts the redexes that belong to the "families" [14] of redexes contracted by P.

Corollary 3.1 (1) (**Erasure Lemma**) Let R be an HRPS. Then $IE_R(s,t)$ iff there is an R_f-reduction $P : t \twoheadrightarrow e$ such that all P-descendants of s in t' are $\underline{S}$-inessential.

(2) Let $t \twoheadrightarrow e$ in R_f. If $ES_R(s,t)$, then s has an R-essential descendant in e.

Lemma 3.10 Let $s = (s_1//t_1, \ldots, s_n//t_n)t$, let $t \prec\!\!\prec_R s$, and let s' and t' be any corresponding subterms of s and t that are outside of replaced subterms. Then $IE_R(s', s) \Rightarrow IE_R(t', t)$.

Proof Let $IE_R(s', s)$. By the Erasure Lemma, there is an R_f-reduction $P : s \twoheadrightarrow e$ such that all P-descendants of s' are $\underline{S}$-inessential. Let $P\|(t) : t \twoheadrightarrow o$ in R_f. Further let $s_i^1, \ldots s_i^{k_i}$ be the enumeration of all P-descendants of s_i from left to right and let $t_i^1, \ldots t_i^{k_i}$ be the enumeration of all $P\|(t)$-descendants of t_i from left to right ($i = 1, \ldots, n$). By Lemma 3.7, o is obtained from e by replacing s_i^j with t_i^j (subterms s_i^j and t_i^j are disjoint, $t_i^j = t_i$, and $s_i \twoheadrightarrow s_i^j$). By Lemma 3.1, there are R-reductions $s_i^j \twoheadrightarrow s_i^{*j}$ and $t_i^j \twoheadrightarrow t_i^{*j}$ such that none of R-inessential free variables of s_i^j and t_i^j that are bound by (outside) $\underline{S}$-operators have descendants in s_i^{*j} and t_i^{*j}. Let e' and o' be obtained from e and o by replacing s_i^j and t_i^j with s_i^{*j} and t_i^{*j}, respectively. Since $t \prec\!\!\prec_R s$, it follows from the conditions (a)-(b) in the Definition 2.2 that $o \prec_{R\cup\underline{S},R} e$; in particular, $o \prec_{\underline{S},R} e$. Hence, it follows from $EBV_{\underline{S},R}(s_i^j) = EBV_{\underline{S},\emptyset}(s_i^{*j}) = BV_{\underline{S}}(s_i^{*j})$ and $EBV_{\underline{S},R}(t_i^j) = EBV_{\underline{S},\emptyset}(t_i^{*j}) = BV_{\underline{S}}(t_i^{*j})$ that $o' \prec_{\underline{S},\emptyset} e'$, i.e., $o' \prec_{\underline{S}} e'$. Since $BV_{\underline{S}}(s_i^{*j}) \subseteq BV_{\underline{S}}(s_i^j)$, we have also that $e' \prec_{\underline{S}} e$. Thus, by the Replacement Lemma, all descendants of s' in e' are $\underline{S}$-inessential and hance all descendants of t' in o' are $\underline{S}$-inessential. Hence, again by the Erasure Lemma, $IE_R(t', t)$.

Corollary 3.2 (**Essential Similarity Lemma**) Let $s = (s_1//t_1, \ldots, s_n//t_n)t$, $t \approx_R s$, and let s' and t' be any corresponding subterms of s and t. Then $IE_R(s', s)$ iff $IE_R(t', t)$.

Corollary 3.3 Corresponding arguments of essentially similar redexes are either both essential or both inessential.

Lemma 3.11 Let $t \xrightarrow{w} s$ in R and let $v \subseteq s$ be a residual of a redex $u \subseteq t$. Then $u \overset{r}{\approx}_R v$.
Proof It is easy to see that if $u \subseteq w$, then $v = u\theta$ for some substitution θ; no free variables are bound in substituted subterms after the substitution. Therefore, it follows from the Essential Similarity Lemma that in this case $u \overset{r}{\approx}_R v$. If u and w do not overlap, then $u = v$. Otherwise, if $w \subseteq u$, then $u \overset{r}{\approx}_R v$ follows from Lemma 3.2.

Lemma 3.12 Let $e_0 \subseteq s_0 \subseteq t_0$, $P : t_0 \xrightarrow{u_0} t_1 \xrightarrow{u_1} \ldots \to t_n$, and let $s_{i+1} \subseteq t_{i+1}$ be a u_i-descendant of $s_i \subseteq t_i$ $(i = 0, \ldots, n-1)$. Then there is a reduction $Q : o_0 = s_0 \xrightarrow{v_0} o_1 \xrightarrow{v_1} \ldots \to o_n$ such that $s_i = o_i\theta_i$ for some substitution θ_i and the descendants of e_0 in s_i and o_i are corresponding occurrences.
Proof By induction on $|P|$. The case $|P| = 0$ is obvious. Suppose that $o_k\theta_k = s_k$. If u_k and s_k do not overlap, then we take $v_k = \emptyset$ and $\theta_k = \theta_{k+1}$ (since $s_k = s_{k+1}$). If u_k is in a substituted subterm of s_k, then again we take $v_k = \emptyset$, and take a corresponding substitution θ_{k+1}. If u_k contains s_k, then again $v_k = \emptyset$ and $\theta_{k+1} = \theta_k\theta$ for some substitution θ. Finally, if $u_k \subseteq s_k$ and is outside of the substituted subterms, then we take as v_k the corresponding redex of u_k in o_k, and take $\theta_k = \theta_{k+1}$.

Lemma 3.13 (1) Each outermost redex in a term t is essential.
(2) $ES(e, t)$ iff $ES(s, t)$ and $ES(e, s)$.
(3) $IE(e, t)$ iff e is in an inessential argument of an essential redex u of t.
Proof (1) From Definition 3.2.
(2) ($\Rightarrow$) From Definition 3.2. ($\Leftarrow$) By Lemma 3.12, $IE(e, t)$ and $ES(s, t)$ would imply $IE(e, s)$, a contradiction.
(3) ($\Rightarrow$) Since $IE(e, t)$, e is inside an outermost, hence essential, redex. Let u be the innermost essential redex containing e in an argument s. Then $ES(s, u)$ would imply by (2) that $IE(e, s)$, and the outermost redex of s that contains e would be essential in t by (2), contrary to the choice of u. ($\Leftarrow$) By (2).

Lemma 3.14 Let R be an HRPS, R' be a subsystem of R, u and v be R-redexes, $u \overset{r}{\prec\!\!\prec}_{R'} v$, $u \xrightarrow{u} o$, and $v \xrightarrow{v} e$. If there is an R'-essential R-redex w in o created by u, then e contains an R'-essential R-redex w', created by v, such that $w \overset{r}{\prec\!\!\prec}_{R'} w'$.
Proof Let o' and e' be contracta of u and v in R_f. Then e' is obtained from o' by replacing the descendants of arguments of u with corresponding descendants of arguments of v. Since the replaced

subterms do not contain $\underline{S}$-redexes (and R'-essentiality coincides with $R' \cup \underline{S}$-essentiality), it follows from conditions (a)-(b) of Definition 2.2 that $o' \prec\!\!\prec_{\underline{S} \cup R'} e'$. Let w_1 be the ancestor of w in o' and w_2 be the corresponding redex of w_1 in e'. By Lemma 3.10 and $o' \prec\!\!\prec_{\underline{S} \cup R'} e'$, $ES_{\underline{S} \cup R'}(w_1, o')$ implies that $ES_{\underline{S} \cup R'}(w_2, e')$. Therefore, w_2 has an R'-essential residual w' in e' (where again R'-essentiality coincides with $R' \cup \underline{S}$-essentiality).

Let $w_1 = \sigma x_1 \dots x_k o_1 \dots o_m$ and $w_2 = \sigma y_1 \dots y_k e_1 \dots e_m$. Let o_j be in the scope of σ and x'_i be an occurrence of x_i in o_j that is $R' \cup \underline{S}$-essential in o_j. If x'_i is outside the replaced subterms, then its corresponding occurrence of y_i in e_j is $R' \cup \underline{S}$-essential in e_j by Lemma 3.10. If x'_i is in a replaced subterm t^*, then, by Lemma 3.13, t^* is $R' \cup \underline{S}$-essential in o_j, its corresponding replaced subterm s^* is $R' \cup \underline{S}$-essential in e_j by Lemma 3.10, and, again by Lemma 3.13, each occurrence of y_i in s^* that is $R' \cup \underline{S}$-essential in s^* (there exists one because $o' \prec\!\!\prec_{\underline{S} \cup R'} e'$) is $R' \cup \underline{S}$-essential in e_j as well. Thus $w_1 \overset{r}{\prec\!\!\prec}_{\underline{S} \cup R'} w_2$. Hence, by Lemma 3.11, $w \overset{r}{\prec\!\!\prec}_{R'} w'$ (since, in e and o, R'-essentiality coincides with $R' \cup \underline{S}$-essentiality).

Corollary 3.4 Let R be an HRPS, $u \overset{r}{\approx}_R v$, $u \xrightarrow{u} o$, and $v \xrightarrow{v} e$. Then there is an R-essential redex w in o created by u iff there is an R-essential redex in e, R-essentially similar to w, created by v.

Remark 3.2 If $s = (s_1//t_1, \dots, s_n//t_n)t$ and $s \approx t$, then it follows from the Essential Similarity Lemma that corresponding redexes of s and t (that are outside of the replaced subterms) are essentially similar (when replaced subterms are specified to be the arguments of the redexes). The converse is not true in general: consider the redexes $u = \sigma x g(y, x)$ and $v = \sigma x g(x, x)$ in a system $R = \{\sigma a A \to \dots\}$; then $u \not\approx_R v$ when $v = (x//y)u$, while $u \overset{r}{\approx}_R v$. However, it follows easily from Lemma 3.13 and Corollary 3.2 that the Essential Similarity Lemma remains valid if the condition $t \approx s$ is replaced by a weaker condition that, for any corresponding redexes w and w' in t and s, $w \overset{r}{\approx} w'$. Similar remark is valid for Lemma 3.10.

4. Decidability of weak normalization in HRPSs

In this section we further study (only) HRPSs. In particular, we construct a decision algorithm for weak normalization. The following definition is crucial; its correctness follows from Corollaries 3.3 and 3.4.

Definition 4.1 (1) We call a sequence of EC-rules $(r_0, ECS(r_0)), (r_1, ECS(r_1)), \dots$ an *essential $(r_0, ECS(r_0))$-chain* if an essential $(r_{i+1}, ECS(r_{i+1}))$-redex is created by contraction of any $(r_i, ECS(r_i))$-redex. For any $(r_0, ECS(r_0))$-redex u, we also call an essential $(r_0, ECS(r_0))$-chain an *essential u-chain*.

(2) Let $(r, ECS(r))$ be an EC-rule and u be an $(r, ECS(r))$-redex. We call the sequence of numbers of essential arguments of u the *essentiality indicator* of $(r, ECS(r))$ and of u.

Lemma 4.1 A term t is weakly normalizable iff any essential chain of any essential redex in t is finite.

Proof ($\Rightarrow$) Consider an infinite essential chain $(r_0, ECS(r_0)), (r_1, ECS(r_1)), \dots$ of an essential redex in t. We show by induction on i that $(\alpha)_i$: for any reduction $P : t = t_0 \xrightarrow{u_0} t_1 \xrightarrow{u_1} \dots$ and for each $i < |P|$ there is a k_i such that t_i contains an essential $(r_{k_i}, ECS(r_{k_i}))$-redex. $(\alpha)_0$ is obvious. If u_i is an essential $(r_{k_i}, ECS(r_{k_i}))$-redex, then it follows from Lemmas 3.2 and 3.13.(2) that t_{i+1} contains an essential $(r_{k_i+1}, ECS(r_{k+1}))$-redex. Otherwise, by Lemmas 3.2 and 3.11, any essential $(r_{k_i}, ECS(r_{k_i}))$-redex in t_i has an essentially similar essential residual in t_{i+1}. Thus $(\alpha)_i$ holds for all i, and t is not normalizable. ($\Leftarrow$) For any $(r, ECS(r))$-redex u, let the *weight* of u be the length of a longest essential $(r, ECS(r))$-chain. Since the contraction of any essential redex creates essential redexes only with smaller weights, a reduction that in each step contracts an innermost redex among essential ones with maximal weight terminates in a normal form.

Lemma 4.2 Let t be a term in an HRPS R and essentiality indicators of all EC-rules in R be known. Then one can find all inessential subterms in t using the following

Algorithm 4.1 Choose in t an innermost redex; find its essential characteristic system (which coincides with its characteristic system); underline its inessential arguments; and mark the redex itself. Now choose in t an unmarked redex that only contains marked redexes; find its essential characteristic system (only occurrences that are in the underlined subterms are inessential); underline its inessential arguments; mark the redex itself; and so on, as long as possible. Then exactly occurrences that are in underlined subterms are inessential in t.

Proof From Lemma 3.13.

Definition 4.2 Let $(r, ECS(r))$ be an EC-rule, let $r : t = \sigma a_1 \dots a_n A_1 \dots A_m \to B$, and let θ be an admissible assignment such that $A_i\theta$ is in R-normal form and $ECS(t\theta) = ECS(r)$. Then we call $r\theta : t\theta \to s\theta$ a *trivial* $ECS(r)$*-instance* of r.

Lemma 4.3 Let $(r_1, ECS(r_1)), \dots, (r_l, ECS(r_l))$ be all EC-rules in an HRPS R and $r_i\theta_i : t_i\theta \to s_i\theta$ be a trivial $ECS(r_i)$-instance of $r_i : t_i \to s_i$ $(i = 1, \dots, l)$. Then the essentiality indicators of the EC-rules in R can be found using the following

Algorithm 4.2 Find, for each i, all arguments of $t_i\theta_i$ that do not have descendants in $s_i\theta_i$. Let the corresponding arguments of any $(r_i, ECS(r_i))$-redex be 0-*inessential*. Let the 0-*essentiality indicator* of $(r_i, ECS(r_i))$ be the list of numbers of arguments of any $(r_i, ECS(r_i))$-redex that are not 0-inessential. Apply Algorithm 4.1 to the right-hand sides of all EC-rules of R using 0-essentiality indicators of EC-rules instead of essentiality indicators. Let the 1-*inessential* arguments of $t_i\theta_i$ be all arguments whose descendants in $s_i\theta_i$ are in underlined subterms $(i = 1, \dots, l)$. Let the corresponding arguments of any $(r_i, ECS(r_i))$-redex be 1-*inessential*, and let the 1-*essentiality indicator* of $(r_i, ECS(r_i))$ be the list of numbers of arguments of any $(r_i, ECS(r_i))$-redex that are not 1-inessential. Apply again Algorithm 4.1 to the right-hand sides of all EC-rules of R using 1-essentiality indicators of EC-rules instead of essentiality indicators; and so on. The algorithm stops after n_0 steps if n_0-essentiality indicator of each EC-rule in R coincides with its $(n_0 - 1)$-essentiality indicator. Let the n_0-inessential arguments of R-redexes be *inessential**. Then the essentiality indicator of $(r_i, ECS(r_i))$ coincides with the list of non-inessential* arguments of $t_i\theta_i$.

Proof An easy induction on n shows that n-inessential arguments of each redex are inessential; the case $n = 0$ follows from Corollary 3.3, and the induction step follows from Lemmas 4.2 and 3.2. Now let us prove by induction on k that if, for an EC-rule $(r_i, ECS(r_i))$, there is an $(r_i, ECS(r_i))$-redex u whose j-th argument s_j (is inessential and) does not have descendants under some reduction P with a length $|P| \leq k$, then the j-th argument e_j of $t_i\theta_i$ is inessential*. The case $k = 1$ is obvious. So suppose $P : u \xrightarrow{u} o \twoheadrightarrow e$. Let s'_j be a descendant of s_j in o, and let v be the minimal redex that contains s'_j and has a descendant in e. Suppose that v is an $(r_m, ECS(r_m))$-redex and s'_j is in its p-th argument. By Lemma 3.12, there is a reduction Q starting from v with a length less than k, such that the p-th argument of v does not have Q-descendants. Hence, by the induction assumption, the p-th argument of $t\theta_m$ is inessential*. Similarly, any descendant of s_j in o is in an inessential* argument of some redex. Since $CS(t_i\theta_i) = ECS(t_i\theta_i) = ECS(u) \subseteq CS(u)$, for any descendant e'_j

of e_j there is a descendant s_j^* of s_j in o that is not in an inessential argument of a non-created redex and such that, for any $q = 1, \ldots, l$, e_j' is in an argument of an $(r_q, ECS(r_q))$-redex iff s_j^* is in the corresponding argument of a created $(r_q, ECS(r_q))$-redex. Hence, any descendant of e_j in $s_i\theta_i$ is in an inessential* argument of some redex, and e_j itself is inessential*.

Algorithm 4.2 can be illustrated by the following example.

Example 4.1 Let us consider an HRPS R with the following rules:

$$\sigma a(A, B) \to (\epsilon aA/a)B$$

$$\delta aA \to \sigma a(A, f(A))$$

$$f(A) \to g(A, A)$$

$$g(A, B) \to const$$

where a is an object metavariable, A, B are term metavariables, f, g are function symbols, σ, ϵ, and δ are quantifier signs with arities $(1, 2)$, $(1, 1)$, and $(1, 1)$, and binding scopes $(1, 2)$, $(1, 1)$, and $(1, 1)$, respectively. The rule for σ has four trivial ECS-instances $r_1 : \sigma x(x, x) \to (\epsilon xx/x)x = \epsilon xx$, $r_2 : \sigma x(x, y) \to (\epsilon xx/x)y = y$, $r_3 : \sigma x(y, x) \to (\epsilon xy/x)x = \epsilon xy$, and $r_4 : \sigma x(y, y) \to (\epsilon xy/x)y = y$ with $ECSs$ $\{(a, A), (a, B)\}, \{(a, A)\}, \{(a, B)\}$, and $\{\}$, respectively. Similarly, we can choose trivial ECS-instances of the rules for δ, f, and g: $r_5 : \delta xx \to \sigma x(x, f(x))$, $r_6 : \delta xy \to \sigma x(y, f(y))$, $r_7 : f(x) \to g(x, x)$, and $r_8 : g(x, y) \to const$. The result of Algorithm 4.2 is then as follows:

$$r_1 : \sigma x(x, x) \to (\epsilon xx/x)x = \epsilon xx$$

$$r_2 : \sigma x(\overbrace{x}^{0}, y) \to (\epsilon xx/x)y = y$$

$$r_3 : \sigma x(y, x) \to (\epsilon xy/x)x = \epsilon xy$$

$$r_4 : \sigma x(\overbrace{y}^{0}, y) \to (\epsilon xy/x)y = y$$

$$r_5 : \delta x \overbrace{x}^{2} \to \sigma x(\underline{x}_{2,3}, f(\underline{x}_{2,3}))$$

$$r_6 : \delta x \overbrace{y}^{2} \to \sigma x(\underline{y}_{1,2,3}, f(\underline{y}_{2,3}))$$

$$r_7 : f(\overbrace{x}^{1}) \to g(\underline{x}_{1,2,3}), \underline{x}_{1,2,3}))$$

$$r_8 : g(\overbrace{x}^{0}, \overbrace{y}^{0}) \to const$$

where $\overbrace{\quad}^{i}$ indicates that corresponding subterms are i-inessential and subscripts in underlined subterms indicate that the underlining was made while running Algorithm 4.1 for the i-th time. The 3-essentiality indicators of all the EC-rules in R coincide with their 2-essentiality indicators, so it is enough to run Algorithm 4.1 three times. Note that, for example, the ECS of the right-hand side $\sigma x(x, f(x))$ of r_5 used while running Algorithm 4.1 for the second time is a proper subset of ECS of $\sigma x(x, f(x))$ used while running Algorithm 4.1 for the first time, and this makes it possible to determine that the argument x of the left-hand side δxx of r_5 is 2-inessential, and hence inessential. By Lemma 4.3, essentiality indicators of EC-rules in R coincide with the numbers of non-overbraced arguments of corresponding trivial instances.

Theorem 4.1 Weak normalization is decidable in HRPSs.
Proof From Lemmas 4.1, 4.2, and 4.3.

Corollary 4.1 HRPSs do not have full computational power.
Proof The existence of an interpreter for, say λ-calculus, in an HRPS would imply decidability of weak normalization for λ-terms, which is not valid [3].

5. Persistent systems

Without restricting the class of OCRSs we can assume that in right-hand sides of rewrite rules the last argument of each metasubstitution is a term-metavariable or a metasubstitution. For example, we can replace the metasubstitution $f((B/a)g(A))$ by the equivalent metasubstitution $f(g((B/a)A))$. Using this convention, we can define the following persistent systems.

Definition 5.1 Let R be an orthogonal CRS.
(1) Let $t \xrightarrow{u} s$, let $t \to t' \twoheadrightarrow s$ be its expansion, let $o \subseteq t'$ be the contractum of u in R_f, and let v be a new redex in s. We call v *generated* if v is a residual of a redex w of t' whose pattern is in the pattern of o. We call v *uniformly generated* if the pattern of w is in the pattern of o and is not inside an $\underline{S}$-redex. We call v *quasi-generated* if any of its pattern-subterms is a descendant of a pattern-subterm of o that is not an $\underline{S}$-redex.
(2) We call R respectively *persistent* (PCRS), *uniformly persistent*, or *quasi-persistent* if, for any R-reduction step, each created redex is generated, uniformly generated, or quasi-generated.
(3) We call R *strongly persistent* if R_{fS} is persistent.

A quasi-persistent CRS $R = \{r_1 : \sigma aA \to f((d/a)A),\ r_2 : f(d) \to c\}$ that is not persistent was considered in the introduction. The CRS $R_1 = \{\exists aA \to g((\tau aA/a)A), f(\tau ac) \to d\}$, where c and d are constants, is an example of persistent system (no redex creation is possible in R_1) that is not strongly persistent.

The following is a characterization of HRPSs in terms of redex creation.

Proposition 5.1 A non-simple OCRS R is strongly persistent iff it is an HRPS.
Proof If R is not simple, then R_{fS} contains $\underline{S}$-rules. If R is not an HRPS, then there is a redex containing at least two function symbols in its pattern, and it can be created during an appropriate $\underline{S}$-step. (For example, if $u = f(g(a))$ is a redex with f, g in the pattern, then $\underline{S}x\, g(a)\, f(x) \to u$.) Hence any non-simple strongly persistent CRS must be an HRPS. The converse is obvious.

We call a subterm s in t *free* if s is not a proper pattern-subterm of a redex in t. It is easy to see that all free subterms remain free under any reduction in PCRSs. This fails already for quasi-persistent systems: in the quasi-persistent CRS $R = \{r_1 : \sigma aA \to f((d/a)A),\ r_2 : f(d) \to c\}$, the argument x is free in the redex σxx, while its descendant d under the contraction of σxx is a pattern-subterm of the created redex $f(d)$. All properties of similar and essentially similar terms in HRPSs are valid because all subterms are free in HRPSs. Therefore, it is not difficult to check that this properties remain valid for all PCRSs if in the definition of similarity and essential similarity one requires the replaced subterms to be free. Since in PCRSs arguments of all redexes are free, one can analogously define essentiality indicators and essential chains for EC-rules in PCRSs. Algorithms 4.1 and 4.2 work also in PCRSs. Hence we have the following theorem.

Theorem 5.1 Weak normalization is decidable in persistent CRSs.

Lemma 5.1 Let R be a persistent CRS, $u \overset{r}{\prec} v$, $u \xrightarrow{u} o$, and $v \xrightarrow{v} e$. If there is a redex w in o generated by u, then e contains a redex w', generated by v, such that $w \overset{r}{\prec} w'$.
Proof The proof of Lemma 3.14 remains valid (after some minor changes) for the case of persistent CRSs. Hence it is enough to take $R' = \emptyset$.

Corollary 5.1 Let u and v be similar redexes in a persistent CRS. Then u and v generate similar redexes.

Definition 5.2 We call a sequence of C-rules $(r_0, CS(r_0)), (r_1, CS(r_1)), \ldots$ an $(r_0, CS(r_0))$-*chain* if an $(r_{i+1}, CS(r_{i+1}))$-redex is generated by contraction of any $(r_i, CS(r_i))$-redex $(i = 0, 1, \ldots)$. For any $(r_0, CS(r_0))$-redex u, we also call an $(r_0, CS(r_0))$-chain a *u-chain*.

Lemma 5.2 A term t in a PCRS R is strongly normalizable iff all chains of redexes in t are finite.
Proof $(\Rightarrow)$ Immediate. $(\Leftarrow)$ Let $P : t = t_0 \xrightarrow{v_0'} t_1 \xrightarrow{v_1'} \ldots$ be an infinite reduction. Let us call a redex $u_n' \subseteq t_n$ a *son* of $u_0' \subseteq t_0$ if there are redexes $u_i' \subseteq t_i$ $(i = 1, \ldots, n-1)$ such that either u_{i+1}' is a residual of u_i' or $u_i' = v_i'$ and u_{i+1}' is generated by u_i. Since in P only redexes from t and their sons are contracted, there is at least one redex $u_0 \subseteq t_0$ such that infinitely many sons of u_0 are contracted in P. Let v_0 be a residual of u_0 that is contracted in P and infinitely many sons of which are contracted in P. Then v_0 generates a redex u_1 whose infinitely many sons are contracted in P. By analogy, u_1 has a residual v_1 contracted in P that generates a redex u_2 whose infinitely many sons are contracted in P, and so on. Obviously, $v_0 \overset{r}{\prec} u_0$. Thus by Lemma 5.1, u_0 generates a redex w_1 such that $u_1 \overset{r}{\prec} w_1$. Further, $v_1 \overset{r}{\prec} u_1 \overset{r}{\prec} w_1$. Thus, again by Lemma 5.1, w_1 generates a redex w_2 such that $u_2 \overset{r}{\prec} w_2$. By analogy we can show that w_2 generates a redex w_3 such that $u_3 \overset{r}{\prec} w_3$, and so on. Thus, u_0 has an infinite chain.

Corollary 5.2 Strong normalization is decidable in persistent CRSs.

Obviously, a PCRS R is weakly (resp. strongly) normalizing iff all essential chains (resp. all chains) in R are finite. Thus it is decidable whether R is weakly (strongly) normalizing.

Remark 5.1 Let R be a persistent CRS and R' be a subsystem of R. We call an R-redex u *$ECS_{R'}$-complete* if each binding variable of a quantifier belonging to the pattern of u has an R'-essential free occurrence in all arguments that belong to its binding scope. It is easy to check that $ECS_{R'}$-complete redexes generate $ECS_{R'}$-complete redexes (this follows from conditions (a)-(b) of Definition 2.2 and Lemma 3.13). By Lemma 3.14, if u and v are weakly similar redexes such that $ECS_{R'}(u) \subseteq ECS_{R'}(v)$ and u generates a redex u', then v generates a redex v' such that $ECS_{R'}(u') \subseteq ECS_{R'}(v')$. Therefore, any R-essential chain (resp. any chain) of a redex w is a subsequence of an R-essential chain (resp. chain) of an ECS_R-complete (resp. $ECS_\emptyset$-complete) redex that is weakly similar to w. Thus, in order to establish weak (strong) normalization of a persistent CRS, it is enough to check essential chains (chains) of ECS_R-complete ($ECS_\emptyset$-complete) rules only.

6. Optimal normalization in persistent CRSs

Theorem 6.1 Let t be a term in a uniformly persistent CRS. Contraction of innermost essential redexes gives a reduction of t to normal form with the least number of steps, whenever the normal form exists.
Proof Let $P : t = t_0 \xrightarrow{u_0} \ldots \to t_n$ be a normalizing reduction starting from t and $Q : t = s_0 \xrightarrow{v_0} s_1 \xrightarrow{v_1} \ldots$ be an innermost essential reduction. It is enough to assign a number $n_i < n$ to each $i < |Q|$ in

such a way that $i \neq j$ implies $n_i \neq n_j$. We show by induction on i that there is a number $n_i < n$ such that $ES(u_{n_i}, t_{n_i})$, $u_{n_i} \stackrel{r}{\approx} v_i$, and $i \neq j$ implies $n_i \neq n_j$.

(1) Let $i = 0$. Since v_0 is essential in t and t_n is a normal form, it follows from Corollary 3.2 that at least one essential residual of v_0 is contracted in P. Thus we can take as n_0 a number such that u_{n_0} is an essential residual of v_0. By Lemma 3.11, $v_0 \stackrel{r}{\approx} u_{n_0}$.

(2) Suppose that, for each $i < m$, we have already defined $n_i < n$ such that $ES(u_{n_i}, t_{n_i})$ and $u_{n_i} \stackrel{r}{\approx} v_i$, and assume that $i = m$. Consider two possible cases: (a) v_m is a residual of a redex $v' \subseteq t$. It follows from $ES(v_m, s_m)$ and Lemma 3.2 that $ES(v', t)$. Thus, as before, there is a number n_m such that u_{n_m} is an essential residual of v', and $v_m \stackrel{r}{\approx} u_{n_m}$ by Lemma 3.11. (b) There is a number k such that v_m is a residual of a redex $v^* \subseteq s_k$ generated by v_{k-1}. By Lemma 3.2, $ES(v_m, s)$ implies $ES(v^*, s_k)$. By the induction assumption, $u_{n_{k-1}}$ is essential and $u_{n_{k-1}} \stackrel{r}{\approx} v_{k-1}$. Let $l = n_{k-1} + 1$. By Corollary 3.4, there is an essential redex $w^* \subseteq t_l$ generated by $u_{n_{k-1}}$ that is essentially similar to v^*. Hence at least one essential residual of w^* is contracted in P, i.e., there is a number n_m such that u_{n_m} is an essential residual of w^*. Thus $ES(u_{n_m}, t_{n_m})$ and, by Lemma 3.11, $u_{n_m} \stackrel{r}{\approx} v_m$. Hence n_i can be constructed for each $i < |Q|$. Since Q is innermost essential, any essential redex in s_i has at most one residual contracted in Q. Hence, $i \neq j$ implies $n_i \neq n_j$ and this completes the proof.

Thus, one can construct optimal sequential normalizing reductions in uniformly persistent CRSs. For PCRSs in general, sharing of redexes of "the same origin" [14] is necessary. For example, one can check that all normalizing reductions starting from a term $\sigma x f(x)$ in the following PCRS $R = \{\sigma a A \to (\varepsilon a A/a)A, \varepsilon a A \to (\tau a A/a)A, f(A) \to g(A, A)\}$ must contract two copies of at least one generated redex. Optimal implementation of PCRSs is possible in the framework of Interaction Systems [2].

7. Further work

We believe that the Essential Similarity Lemma can be generalized to the case of the λ-calculus. The replaced subterms must be *independent*: a subterm is independent if it is free and remains free under any reduction; moreover, redex creation is not possible inside an independent subterm during contraction of a redexes outside it. In this case, independent subterms are indeed independent in the sense that their computation can be made in parallel.

One can also define a class of constructor CRSs (pattern-matching definitions) that are persistent and enjoy the decidability of weak normalization.

Forthcoming papers will be devoted to a classification of OCRSs based on the patterns of redex creation and to a study of normalization and perpetuality behaviour in several subclasses. In this context, it is interesting to investigate whether *inside creating* systems, where each created redex is inside the contractum of the creating redex, have full computational power. In inside-creating systems, all outermost redexes are essential and independent, hence any outside-in strategy is normalizing and non-overlapping redexes can be computed independently.

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