The Conflict-free Reduction Geometry *

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Abstract

We investigate mutual *dependencies* of subexpressions of computable expressions in orthogonal rewrite systems, and identify conditions for their *independent* concurrent computation. To this end, we introduce concepts familiar from ordinary Euclidean Geometry (such as *basis*, *projection*, *distance*, etc.) for reduction spaces. We show how a basis at an expression can be constructed so that any reduction starting from that expression can be decomposed as the sum of its projections on the axes of the basis. To make the concepts computationally more relevant, we relativize them w.r.t. *stable* sets of results (such as the set of normal forms, head-normal forms, and weak-head-normal forms, in the λ-calculus), and show that an optimal concurrent computation of an expression w.r.t. $\mathcal{S}$ consists of optimal computations of its $\mathcal{S}$-independent subexpressions. All these results are obtained in the framework of *Deterministic Family Structures*, which are Abstract Reduction Systems with axiomatized *residual* and *family* relations on redexes, that model all orthogonal rewrite systems.

1 Introduction

Efficient evaluation of expressions benefits from concurrent evaluation of subexpressions. In computation in general, it is normal that intermediate results of

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different subexpressions are used by other subexpressions, and contribute to creation of new computable subexpressions. In concurrent languages like the π-calculus [32] this is expressed explicitly by value-passing, while in sequential languages computations in different subexpressions can only *interact* by common creation of new redexes. Our aim in this paper is to give a formal numerical characterization of *dependencies* of subexpressions of an expression (or subprograms of a modular program), and in particular to identify conditions for *independent* evaluation of subexpressions. Computation of independent subexpressions can be conducted concurrently, in isolation from computations elsewhere in the expression, and the results can then be combined to yield the final result. We restrict our attention to functional languages, and consider their operational model – orthogonal rewrite systems – of which the λ-calculus [2] is the prime example. We believe that our results can be generalized to the non-orthogonal case and cover the concurrent languages as well.

The idea we want to pursue is very simple and natural, and the concepts we introduce have their counterparts in ordinary Euclidean Geometry, although there will be some differences. For expository purposes, let us assume first that the given orthogonal rewrite system is *linear* – there is neither duplication nor erasure of redexes. The main analogy is the following: in a Euclidean 3-dimensional space, one can decompose a vector as the sum of its projections on the axes X, Y and Z, which form a Euclidean basis. Similarly, we can construct a basis at any expression t, consisting of *independent* reductions P_i starting from t, such that any reduction P starting from t can be decomposed as the sum of its projections on P_i, up to *Lévy* or *permutation-equivalence*, $\approx_L$. Here the 'sum' operation is the least upper bound operation, $\sqcup$, in the reduction space ordered by Lévy's embedding relation $\trianglelefteq_L$, and $\approx_L = \trianglelefteq_L \cap \trianglerighteq_L$ [30]. P_i and P_j are independent if no finite initial parts of them can *interact*, i.e., by common creation of a new redex. In the bases that we construct, every reduction P_i is a maximal reduction *internal* to a redex-set U_i in t, i.e, P_i contracts only residuals of redexes in U_i and newly created redexes (which we call U_i-*redexes*); every U_i is *independent*, i.e., no reduction internal to U_i can interact with a reduction internal to the complement $\overline{U_i}$ of U_i, which consists of redexes of t not in U_i; U_i are pairwise non-overlapping, and cover all redexes of t.

Further, the *distance* $\|P, Q\|$ between co-initial reductions P, Q is the number of their 'different' steps, and characterizes 'how far apart' the reductions have progressed. Here 'different steps' means that they cannot be related by the *zig-zag* relation [30], so they are in different *zig-zag families*. (We recall that zig-zag is simply the transitive and symmetric closure of the residual relation on redexes with creation histories.) When both P and Q are finite, $\|P, Q\|$ coincides with the minimal number of reduction steps needed to reach a common reduct from the endpoints of P and Q. This is different from the Euclidean measure of distance. For example, in the simplest case, if two vectors $\vec{P}$ and $\vec{Q}$ are orthogonal (say parallel to axes X and Y respectively), then the distance is $\|\vec{P},\vec{Q}\| = \sqrt{|\vec{P}|^2 + |\vec{Q}|^2}$, while the distance between reductions P and Q that contract redexes in different families is $\|P,Q\| = |P| + |Q|$. However, this is because the Euclidean space is continuous and allows 'shortcuts'. If we were to allow joining of the endpoints of the vectors $\vec{P}$ and $\vec{Q}$ only by moves parallel to X and Y, then we would get the same distance measure as for reductions.

Finally, the *independence degree* of a redex set U of an expression t is the length of a shortest reduction P internal to U such that there is a reduction Q internal to $\overline{U}$ that interacts with P, and is ∞ otherwise. So at least $|P|$ steps can be performed in U independently from the rest of the computation, after which results of computing U and $\overline{U}$ must be combined in order for the computation to proceed 'as concurrently as possible'. Note that if and only if U is independent, its independence degree is ∞.

These concepts can very naturally be explained in terms of *Prime Event Structures* (PES) [37], which, in the conflict-free case in which we are interested, are simply *event* sets E partially ordered by a *causal dependency* relation $\leq$, such that every event $e \in E$ can only dominate a finite number of others. Computations in a linear orthogonal rewrite systems R are interpreted as left-closed sets of events (i.e., closed under $\leq$), called *configurations*, in the PES $\mathcal{E} = (E, \leq)$ whose events correspond to (the zig-zag classes of) redexes in R. Those configurations $X_i \subseteq E$ that are closed under $\geq$ are *independent*, as they correspond to independent reductions in R. Further, if $\{X_i \mid i \in I\}$ are disjoint independent sets covering E, they form a *basis* for E, as for any configuration α, $\alpha = \cup_{i\in I} \alpha \cap X_i$. Here $\alpha \cap X_i$ is the *projection* of α on X_i, and coincides with the *restriction* of α to the set X_i^0 of all initial (i.e., minimal w.r.t. $\leq$) events of X_i. The set $\{X_i^0\}_{i\in I}$ is an *independent covering* of the set of initial events of E. Further, the *distance* between configurations α and β is defined as the cardinality of $\alpha \cup \beta \setminus \alpha \cap \beta$ (as is usual for sets), and it precisely corresponds to the distance measure for reductions in linear orthogonal rewrite systems – $\|P, Q\| = \|\alpha_P, \alpha_Q\|$, where α_P, α_Q are configurations corresponding to P, Q. The *independence degree* of a set α_0 of initial events is the cardinality of the smallest configuration α such that there exists a configuration β not containing elements of α_0 and an event e such that $\alpha \cup \beta \cup \{e\}$ is a configuration, while neither $\alpha \cup \{e\}$ nor $\beta \cup \{e\}$ are (i.e, α and β both contribute to creation of e, and they interact to create e).

The decomposition property, which is the main goal of our study of the independence concept, immediately breaks down if one wants to decompose any reduction in a duplicating reduction system. For example, consider a λ-term $v = (\lambda x.xx)u$, where u is a β-redex. Then $\{v\}$ and $\{u\}$ form an independent redex-covering of v as u and v cannot commonly create a new redex. The reduction $P : v \xrightarrow{v} uu \xrightarrow{u} ou$, contracting v and then the leftmost residual of u is not Lévy-equivalent to $u \sqcup v$, and the reason is that P contracts only one copy of the duplicated redex u. Note here that v contains u in the argument and can duplicate it. There is no reason why we should not allow the situation where redexes in one independent set duplicate or erase redexes in another independent set. Indeed, such a restriction would yield a trivial concept of independence as in that case (in general) only redex-sets in disjoint subterms may be regarded as independent, and this would result in the loss of efficiency in concurrent evaluation of an expression (if only independent redex-sets are to be evaluated concurrently). For example, consider a Term Rewriting System (TRS) $R = \{f_i(x) \to f_{i+1}(x),\, g_i(x) \to g_{i+1}(x) \,|\, i = 1, \ldots, 5\}$. Then in the expression $f_1(g_1(x))$ we could independently evaluate components $f_1(y)$ and $g_1(x)$, yielding $f_5(y)$ and $g_5(x)$, respectively, and then combine these intermediate results into the final result, $f_5(g_5(x))$, by substituting $g_5(x)$ for y in $f_5(y)$, which is 'quicker' then normalizing $f_1(g_1(x))$ in ten sequential steps.

To recover the decomposition property for duplicating (and erasing) systems, we instead restrict ourselves to *complete family-reductions* (up to $\approx_L$), which are multi-step reductions contracting in each multi-step an entire family of redexes [29,30]. Redexes in a family are redexes of 'the same origin', and according to Lévy's approach to optimal evaluation, these are the redexes that must be shared in a graph implementation of the λ-calculus. Such implementations have indeed been achieved by Lamping [27] and Kathail [9]. Although in general there may be reductions that are not complete family-reductions and that can be decomposed w.r.t. an independent basis, there seems to be no simple characterization of such a class of decomposable reductions independent from the particular rewrite system and the particular basis involved, and we leave such a refinement of our approach for a future work. Furthermore, restriction to complete family-reductions is not a restriction from the computational point of view as such reductions can compute all the kinds of result one is usually interested in in the theory or practice of functional programming, such as normal forms, head-normal forms, or weak-head-normal forms, in the λ-calculus. For example, reductions of term graphs [11] or directed acyclic graphs (DAGs) [31] correspond exactly to complete family reductions on terms, in orthogonal term rewriting systems (while reduction of sharing-graphs correspond to complete family-reductions in the λ-calculus only up to 'book-keeping' steps).

In order to make the introduced concepts computationally more meaningful, we relativize them w.r.t. the semantics one may be interested in. For example, in the λ-calculus, one might be interested in computing normal forms, head-normal forms, weak-head-normal forms, etc. In [4], we have characterized all reasonable sets of finite '(partial) results' as *stable* sets $\mathcal{S}$ of terms, and have shown that w.r.t. stable sets $\mathcal{S}$, $\mathcal{S}$-needed reductions are $\mathcal{S}$-normalizing (i.e., end at a term in $\mathcal{S}$ if the initial term is reducible to a term in $\mathcal{S}$). This allows us to ignore $\mathcal{S}$-unneeded redexes, and for example, we can define P, Q to be $\mathcal{S}$-*independent* if there is no common creation of $\mathcal{S}$-needed redexes. So reductions that interact may be $\mathcal{S}$-independent. This is profitable since redex sets that are not independent may become $\mathcal{S}$-independent, and this allows for finer independent splitting of redex-sets of terms, implying more parallelism in the computation. Indeed, if $\{U_i\}_{i \in I}$ is an $\mathcal{S}$-independent covering of an $\mathcal{S}$-normalizable term t, in a possibly duplicating and/or erasing orthogonal rewriting system, we show that an optimal $\mathcal{S}$-normalizing reduction starting from t (which is an $\mathcal{S}$-needed $\mathcal{S}$-normalizing complete family-reduction) is the sum of $\mathcal{S}$-needed complete family-reductions P_i internal to U_i. Note that P_i are $(U_i, \mathcal{S})$-*fair*, meaning that the final terms of P_i do not contain $\mathcal{S}$-needed U_i-redexes.

To remain as general as possible, and at the same time to avoid syntactic structure of computable expressions (terms, graphs, etc.), which is irrelevant for our purpose, we assume that the rewrite system is given in the form of a *Deterministic Family Structure*, DFS [6]. DFSs are Abstract Reduction Systems (ARSs) with axiomatized residual and family relations, which model all orthogonal rewriting systems and cover all existing concepts of redex-family in the literature [29,30,10,31,1,34]. Important standard results such as the Standardization and Normalization Theorems of Curry and Feys [2], or Lévy's Optimality Theorem [29,30], can be proven using only the abstract framework of DFSs [6,16]. Furthermore, by using the DFS axioms alone, one can interpret DFSs into non-duplicating, also called affine, DFSs with zig-zag as the family relation, by interpreting family-reduction multi-steps in the former as reduction steps in the latter [19,20]. This allows us to reduce studying the decomposition property from DFSs to affine zig-zag DFSs (AZDFSs), and this enormously simplifies the theory of independent computation and the proofs.

Most of the (remaining) technical difficulties come from the erasure of redexes in AZDFSs. To cope with the erasure problems, and to have (most of the) concepts invariant under Lévy-equivalence, we work with *standard* reductions, which in DFSs are reductions in which later steps 'do not erase' the preceding ones (we ignore outside-in and left-to-right order of contraction of redexes, as there are no such concepts in DFSs, and these are actually inappropriate) [16]. Even if restricted to standard reductions, the reduction space of an AZDFS ordered by permutation embedding need not be isomorphic to the domain generated by a PES. Therefore, instead of PESs, we will (by necessity) use event models, namely Deterministic Erasure Event Structures (DEESs), with an axiomatized erasure relation, developed in [18,21], to give more intuitive interpretations of our constructions in terms of events (or equivalently, in terms of *distributedness*). This interpretation also shows that our Geometry metaphor can be seen as a further development of the Event Structure approach to modelling distributedness.

To conclude, we note that in any DFS $\mathcal{F}$, we can define independent redex sets $U_i \subseteq t$ via interpreting $\mathcal{F}$ into the corresponding AZDFS $\mathcal{F}_I$, and compute the sets U_i independently in $\mathcal{F}$ (not necessarily using complete family-reductions, if Lévy-style optimality is not a concern). Indeed, we can easily show that for any set Q_i of $(U_i, \mathcal{S})$-fair reductions in $\mathcal{F}$, $\sqcup_{i \in I} Q_i$ is $\mathcal{S}$-normalizing. Hence our restriction to affine systems when developing the (intermediate) independent decomposition results has no implications for the applicability of our results to duplicating and/or erasing orthogonal systems when one is interested in independent computations of (relative) normal forms.

The paper is organized as follows. In the next section, we recall some theory of DFSs used in this paper. In section 3, we introduce the restriction and projection concepts, study their properties, and prove the *Decomposition Theorem*. In section 4, we define the geometry of orthogonal reduction spaces, and prove the *Independent Decomposition Theorem*. In section 5, we relativize the geometry w.r.t. stable sets of results $\mathcal{S}$, and show that an optimal computation w.r.t. $\mathcal{S}$ can be achieved by combining optimal computations of $\mathcal{S}$-independent redex-sets. Conclusions appear in section 6.

An extended abstract of this work appears as [17].

2 Preliminaries

In this section, we quickly recall Deterministic Family Structures (DFSs) [6,20]. We assume that the reader is familiar with the concepts related to Lévy Permutation equivalence [29,8,30,2]. Related abstract residual models are studied in [36,7,33].

We start by introducing Abstract Reduction Systems (ARSs). Our definition is slightly different from the usual one [26]. We then define *Deterministic Residual Structures* (DRSs) which are ARSs with an axiomatized notion of *residual*. *Deterministic Family Structures* (DFSs) are then defined as DRSs with an axiomatized concept of *redex-family* [29].

Definition 1 An ARS is a triple $A = (Ter, Red, \to)$ where Ter is a set of *terms*, ranged over by t, s, o, e; Red is a set of *redexes* (or *redex occurrences*), ranged over by u, v, w; and $\to$: $Red \mapsto (Ter \times Ter)$ is a function such that for any $t \in Ter$ there is only a finite set of $u \in Red$ such that $\mapsto (u) = (t, s)$, written $t \stackrel{u}{\to} s$. This set will be known as the redexes of term t. Note that $\to$ is a *total* function, so one can identify u with the triple $t \stackrel{u}{\to} s$. [1] A *reduction* is a (finite or infinite) sequence $t \stackrel{u_1}{\to} t_2 \stackrel{u_2}{\to} \ldots$.

[1] Klop's ARSs [26] are pairs $(Ter, \to)$, and cannot distinguish transitions with the same source and target terms.

Notation 2 *Reductions are denoted by P, Q, N. We write $P : t \twoheadrightarrow s$ or $t \overset{P}{\twoheadrightarrow} s$ if P denotes a reduction (sequence) from t to s, and write $P : t \twoheadrightarrow$ if P may be infinite. $P + Q$ denotes the concatenation of P and Q* (when P is finite and its final term coincides with the initial term of Q). *u also denotes the reduction that contracts u. The final term of a finite reduction P is denoted by $ft(P)$. $Q \leq P$ denotes that Q is an initial part of P; if moreover Q is finite, we write $Q \leq_{fin} P$. Finally,* $u \subseteq t$ denotes that u is a member of the redexes of t, and *$U \subseteq t$ denotes that U is a subset of the redexes in t. If $U \subseteq t$, then $\overline{U}$ will denote the* complement *of U, i.e., the set of redexes in t not in U.*

Definition 3 **(1)** A *Deterministic Residual Structure* (DRS) is a pair $\mathcal{R} = (A, /)$, where A is an ARS and $/$ is a *residual* relation on redexes relating redexes in the source and target term of every reduction $t \overset{u}{\to} s \in A$, such that for $v \subseteq t$, the set v/u of *residuals of v under u* is a set of redexes of s; a redex in s may be a residual of only one redex in t under u, and $u/u = \emptyset$. If v has more than one u-residual, then u *duplicates* v. If $v/u = \emptyset$, then u *erases* v. A redex of s which is not a residual of any $v \subseteq t$ under u is said to be *u-new* or *created* by u. The set u/P of residuals of u under any finite reduction P is defined by transitivity.

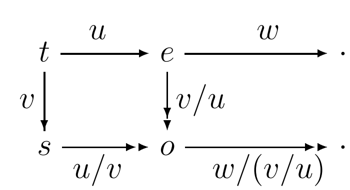
A *development* of $U \subseteq t$ is a reduction $P : t \twoheadrightarrow$ that only contracts residuals of redexes from U; it is *complete* if it is finite and $U/P = \cup_{u \in U} u/P = \emptyset$. Development of $\emptyset$ is identified with the empty reduction. The residual relation satisfies the following two axioms:

[FD] ([7]) All developments are terminating; all complete developments of $U \subseteq t$ end at the same term; and residuals of a redex $v \subseteq t$ under all complete developments of U are the same. Below U will also denote a complete development of $U \subseteq t$.

[weak acyclicity] ([36]) Let $u, v \subseteq t$, let $u \neq v$, and let $u/v = \emptyset$. Then $v/u \neq \emptyset$.

(2) We call a DRS $\mathcal{R}$ *stable* (SDRS) if the following axiom is satisfied:

[stability] If $u, v \subseteq t$ are different redexes, $t \overset{u}{\to} e$, $t \overset{v}{\to} s$, and u creates a redex $w \subseteq e$, then the redexes in $w/(v/u)$ are not u/v-residuals of redexes of s, i.e., they are created by u/v.



(3) We call a DRS $\mathcal{R}$ *non-duplicating* or *affine* if its residual relation is non-duplicating (i.e., a redex may have at most one residual under contraction of another redex). Affine SDRSs will be called ASDRSs.

In a DRS $\mathcal{R}$, the residual relation $/$, *Lévy permutation embedding* $\trianglelefteq_L$, and *Lévy permutation equivalence* $\approx_L$, on finite co-initial reductions are defined exactly as in syntactic orthogonal rewrite systems, see [2,8,30,36]. These concepts naturally extend to infinite reductions as well (except N/P is only defined for finite P), see e.g. [16]. We only recall that the *Strong Church-Rosser* property states that for any co-initial finite reductions P, Q, $P \sqcup Q \approx_L Q \sqcup P$, where $P \sqcup Q = P + Q/P$. Another useful property, the *Cube Lemma*, states that for any finite co-initial reductions P, Q and N, $N/(P \sqcup Q) = N/(Q \sqcup P)$. In the particular, when $N = v$, the Cube Lemma implies that the residuals of v in $ft(Q \sqcup P)$ along $Q \sqcup P$ and $P \sqcup Q$ are the same.

We remark that, for finite co-initial reductions P and Q, P/Q is defined uniquely if we regard it as a *multi-step* reduction, where a multi-step contracts simultaneously a set of redexes in a term. If we want to regard P/Q as a reduction, then it is defined uniquely only up to the particular sequentializations of the corresponding multi-steps. (However, P/Q is defined more precisely than up to $\approx_L$.) It is conventional to switch freely between multi-steps and complete developments, and this does not cause any problems for the developments in this paper.

Definition 4 A *Deterministic Family Structure* (DFS) is a triple $\mathcal{F} = (\mathcal{R}, \simeq, \hookrightarrow)$, where $\mathcal{R}$ is a DRS; a family relation $\simeq$ is an equivalence relation on redexes with *histories*; and $\hookrightarrow$ is the *contribution* relation on co-initial families, defined as follows:

(1) For any co-initial finite reductions P and Q, a redex Qv in the final term of Q (read as v *with history* Q) is called a *copy* of a redex Pu if $P \trianglelefteq_L Q$, i.e., $P + Q/P \approx_L Q$, and v is a Q/P-residual of u; the *zig-zag* relation $\simeq_z$ is the symmetric and transitive closure of the copy relation. The *family* relation $\simeq$ is an equivalence relation among redexes with histories containing $\simeq_z$. A *family* is an equivalence class of the family relation; families are ranged over by $\phi, \psi, \ldots$. $Fam(\ )$ denotes the family of its argument.

(2) Further, $\simeq$ and $\hookrightarrow$ satisfy the following axioms:

[initial] Let $u, v \subseteq t$ and $u \neq v$, in $\mathcal{R}$. Then $Fam(\emptyset_t u) \neq Fam(\emptyset_t v)$, where $\emptyset_t$ is the empty reduction starting from t.

[contribution] $\phi \hookrightarrow \phi'$ iff for any $Pu \in \phi'$, P contracts at least one redex in ϕ.

[creation] Let $e \overset{P}{\twoheadrightarrow} t \overset{u}{\rightarrow} s$, and let u create $v \subseteq s$. Then $Fam(u) \hookrightarrow Fam(v)$, or more precisely, $Fam(Pu) \hookrightarrow Fam((P+u)v)$.

[FFD] (*Finite Family Developments*) Any reduction that contracts redexes of a finite number of families is terminating. [2]

(3) We call a DFS $\mathcal{F}$ a *zig-zag* DFS, ZDFS, if its family relation is the zig-zag $\simeq_z$ (the corresponding contribution relation $\hookrightarrow_z$ is determined by [contribution]). Affine ZDFSs will be called AZDFSs.

All concepts of family for orthogonal reduction systems known to us (such

[2] This axiom is called [termination] in [6].

as [29,30,10,31,1,34]) satisfy our family axioms ([FFD] is not easy to prove for concrete systems, while the other family axioms can easily be verified using the labelling definitions of these family concepts). These axioms allow for example for abstract proofs of Relative Normalization and Optimality Theorems [6,20].

As already mentioned in the introduction, DFSs $\mathcal{F}$ can be interpreted into AZDFSs $\mathcal{F}_I$. Since this interpretation allows us to restrict our investigation into the independence concept to AZDFSs, we very briefly recall the construction from [19,20]. Recall that a *family-reduction* in $\mathcal{F}$ is a multi-step reduction contracting in each multi-step a set of redexes in a single family; and a family-reduction is *complete* if each multi-step contracts all redexes of a single family in the corresponding term [30]. Complete family-reductions in $\mathcal{F}$ are interpreted as reductions in the corresponding AZDFS $\mathcal{F}_I$. Since whether a redex set $U \subseteq s$ is a family depends on the history of redexes in U, the interpretation requires all histories to be co-initial, i.e., an *initial* term is to be fixed in $\mathcal{F}$, and only complete family-reductions relative to that term will have corresponding reductions in $\mathcal{F}_I$. Thus the reduction graph of $\mathcal{F}_I$ consists of (some of the) reducts of the initial term, and the residual in $\mathcal{F}_I$ is induced by that of $\mathcal{F}$.

Theorem 5 ([19,20]) For any DFS $\mathcal{F}$, $\mathcal{F}_I$ is an AZDFS.

Remark 6 *There is actually an isomorphism between complete family-reductions in $\mathcal{F}$ and reductions in $\mathcal{F}_I$. Thus all definitions and results in this paper concerning reductions in* AZDFSs *can be read as definitions or results concerning complete family-reductions in arbitrary DFSs.*

In AZDFSs, any term contains at most one member redex of a family:

Proposition 7 ([19,20]) Let $Pv \simeq_z Qw$ in an AZDFS. Then $v/(Q/P) = w/(P/Q)$. In particular, if $P \approx_L Q$, then $v = w$.

3 Decomposition of Reductions in AZDFSs

In this section, we define the *projection* of a reduction onto another one and its *restriction* to a redex-set, study their properties, and use them to decompose reductions as the sum of their restrictions to non-overlapping redex sets. The projection concept is based on a *relative standardization* algorithm [16].

3.1 Standardization

In this subsection, we define a somewhat non-standard concept of standardization. Firstly, our standard reductions are those in which later steps ‘do not erase' preceding ones. In DFSs we cannot define a 'left-to-right' and/or 'outside-in' concept of standard reduction familiar from the λ-calculus and orthogonal rewriting systems [2,8,25], but we do not need such a standardization concept either. Secondly, we define a *relativized* standardization algorithm which standardizes any reduction Q in an AZDFS w.r.t. any other reduction P, co-initial with Q.

Definition 8 (Standard Reductions [16]) Let a DRS be given.

- Let $U \subseteq t$ and $P : t \twoheadrightarrow$ (so P may be infinite). We call P *external to* U if P does not contract residuals of redexes in U. If $U = \{u\}$, then P is called *external to* u. If $u/P' = \emptyset$ for some $P' \leq_{fin} P$, then we say that u is *erased in* P or is P*-erased*, or that P *erases* u. If P is external to u and erases it, then we say that P *discards* u.
- Let $P : t \twoheadrightarrow$ and $u \subseteq t$. We call u P*-needed* if there is no $Q \approx_L P$ that is external to u, and call it P*-unneeded* otherwise.
- Let $Q : t \twoheadrightarrow$, $P : t \overset{P'}{\twoheadrightarrow} s \twoheadrightarrow$, and $u \subseteq s$. We call u (or rather $P'u$) Q-(un)*needed* if u is Q/P'-(un)needed. We call P Q-(un)*needed* if every redex contracted in P is Q-(un)needed.
- We call P *self-needed* or *standard* if it is P-needed. We write $Q \approx_{STA} P$ if $Q \approx_L P$ and $Q, P \in STA$, where STA denotes the set of all standard reductions. We call N a *standard variant* of P if $P \approx_L N$ and $N \in STA$.

Note that P-neededness does not depend on the choice of a reduction in the class of reductions Lévy-equivalent to P, since $u/P = u/Q$ when $P \approx_L Q$, by the Cube Lemma. The same holds true for P-erasure, but not for the concepts 'external' and 'discards'.

Definition 9 **(1)** Let $P, Q : t \twoheadrightarrow$. The *canonical* P*-needed variant of* Q, written $ST_P(Q)$, is defined as follows: let $v \subseteq t$ be such that it is P-needed and its (not necessarily P-needed) residual is contracted in Q first among residuals of P-needed redexes in t (note that v need not be Q-needed). Then $ST_P(Q) = v + ST_{P/v}(Q/v)$. If there is no such a redex in t, then $ST_P(Q) = \emptyset$.
(2) If $Q = P$, then we refer to the above algorithm as the standardization algorithm; we will write $ST(Q)$ for $ST_Q(Q)$, and call $ST(Q)$ the *canonical standard variant* of Q.

The above algorithm and the Standardization Theorem below differ slightly from the ones in [16] since, unlike [16], here we cannot restrict ourselves to the case when $Q \trianglelefteq_L P$. However, when $Q \trianglelefteq_L P$ (or $Q \approx_L P$), the two algorithms are equivalent.

Lemma 10 If $u \trianglelefteq_L P \trianglelefteq_L Q$ and u is Q-needed, in an AZDFS, then it is P-needed. Consequently, if P is Q-needed, then it is standard.

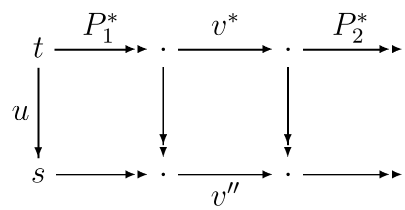
Proof Suppose on the contrary that u is P-unneeded, i.e, there is $P' \approx_L P$

that is external to u. Since u is P-erased, P' discards u, hence there is a $P'' \leq_{fin} P'$ that discards u. But $Q \approx_L P''+Q''$ for some Q'', contradicting Q-neededness of u. Thus u is P-needed, and the rest is immediate from Definition 8.

Lemma 11 Let $t \xrightarrow{u} s$, $P : t \twoheadrightarrow$, and $v' = v/u$, in an AZDFS.

(1) If v is P-unneeded, then so is v'.
(2) ([16]) If moreover $u/P = \emptyset$, then the converse is also true.

Proof We only prove (1). Since v is P-unneeded, there is $P^* \approx_L P$ that is external to v. If on the contrary v' was P-needed, P^*/u would contract a residual v'' of v', and P^* would contract a redex v^*, say $P^* = P_1^* + v^* + P_2^*$, such that $P_1^* v^* \simeq_z \emptyset v$, implying by Proposition 7 that P^* contracts a residual v^* of v – a contradiction.



Note that without the condition $u/P = \emptyset$, Lemma 11.(2) is not valid: Let $t = u = (\lambda x.w)v$ and $P = v \sqcup w$, where $w = Kyx$ and $K = \lambda x.\lambda y.x$. Then v is P-needed, $P/u : Kyv' \xrightarrow{v'} Kyo \twoheadrightarrow y$, and v' is P/u-unneeded.

Theorem 12 (**Standardization**) Let an AZDFS be given.

(1) For any co-initial reductions Q, P, finite or infinite, $ST_P(Q)$ is a P-needed, and standard, reduction such that $ST_P(Q) \trianglelefteq_L Q, P$.
(2) ([16]) Furthermore, if Q is finite, then $Q \approx_L ST(Q)$; otherwise, $Q \approx_L ST(Q)$ need not hold.

Proof We only prove (1). By Definition 9, we have

$$\begin{aligned} ST_P(Q) &= v_0 + ST_{P/v_0}(Q/v_0) \\ &= v_0 + v_1 + ST_{P/(v_0+v_1)}(Q/(v_0+v_1)) \\ &= v_0 + v_1 + v_2 + \dots \end{aligned}$$

where v_i is $P/(v_0 + v_1 + \dots + v_{i-1})$-needed and its residual is contracted in $Q/(v_0 + v_1 + \dots + v_{i-1})$. Denote $N_i = v_0 + v_1 + \dots + v_{i-1}$. Then $v_i \trianglelefteq_L P/N_i$, $v_i \trianglelefteq_L Q/N_i$, $ST_P(Q)$ is P-needed (by Definition 8), and $ST_P(Q) \trianglelefteq_L Q, P$. Hence $ST_P(Q)$ is standard by Lemma 10.

It has been shown in [19,20] that, in AZDFSs, all standard variants of a finite reduction P can be constructed effectively (as P-neededness is decidable and there is only a finite number of such reductions, all of the same length), and that $\approx_{STA}$ is decidable. So finite standard reductions can be used as canonical representatives of their Lévy-equivalence classes (which may have an infinite number of elements). As we have seen above, this is not true for infinite reductions in general, and we need to be careful in our statements and proofs when considering standard variants of infinite reductions.

3.2 Characterizing permutation-equivalence using families

In this subsection, we give a characterization of permutation-equivalence in AZDFSs via families of contracted redexes in corresponding reductions, and use it in the next subsection to give a similar characterization for the relative standardization concept. These characterizations significantly simplify our proof of the Decomposition Theorem – the main result of this section, and allow us to interpret that theorem in DEESs.

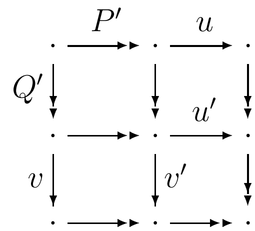
Notation 13 *Below, $FAM(P)$ (resp. $SFAM(P)$) denotes the set of families* of which at least one (resp. P-needed) member redex is *contracted in P, in a DFS. The initial term (from which all histories start) will be clear from the context. For example, if $P : t_0 \xrightarrow{u_0} t_1 \xrightarrow{u_1} \ldots$ is an initial reduction, then $FAM(P) = \cup_i \{Fam(P_i u_i)\}$, where $P_i : t_0 \xrightarrow{u_0} \ldots \xrightarrow{u_{i-1}} t_i$, and if Q is a non-initial part of an initial reduction $N = Q' + Q$, then $FAM(Q)$ is the corresponding subset of $FAM(N)$.*

Lemma 14 Let P and Q be standard co-initial reductions, in an AZDFS. Then $P \approx_L Q$ iff $FAM(P) = FAM(Q)$.

Proof

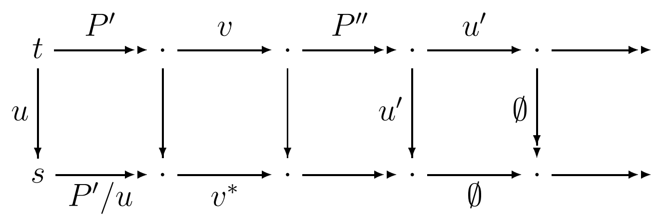
- ($\Rightarrow$) Let w be a contracted redex in P, say $P = P' + w + P''$. Then w is Q/P'-needed. Hence $FAM(P'w) \in FAM(Q/P') \subseteq FAM(Q)$, and the converse is proved similarly.
- ($\Leftarrow$) Suppose on the contrary that $P \not\approx_L Q$, and say $P/Q \neq \emptyset$. Then P contracts a redex u, say $P = P' + u + P''$, such that $u/(Q/P') \neq \emptyset$. Let v be a step in Q, i.e., $Q = Q' + v + Q''$ (see the figure). Then if $u' = u/(Q'/P')$ and $v' = v/(P'/Q')$, we have $u' \neq v'$. Hence, by Proposition 7, $Fam(P'u) = Fam((P' \sqcup Q')u') \neq Fam((P' \sqcup Q')v') = Fam(Q'v)$, i.e.,

$FAM(P) \ni Fam(P'u) \notin FAM(Q)$ – contradiction.



Lemma 15 Let $u \in t \overset{P}{\twoheadrightarrow}$ and let u be P-needed, in an AZDFS. Then $SFAM(P) = SFAM(P/u) \cup \{Fam(\emptyset u)\}$.

Proof Let v be a redex contracted in P that is not a residual of u, say $P = P' + v + P''$, and let $v^* = v/(u/P')$ (see the diagram). Then it is enough to prove that $Fam(P'v) \in SFAM(P)$ iff $Fam((P'/u)v^*) \in SFAM(P/u)$. But the latter is immediate from Lemma 11.



Lemma 16 Let P be a (finite or infinite) reduction in an AZDFS. Then $SFAM(P) = FAM(ST(P))$.

Proof Immediate from Lemma 15 and Theorem 12.(1).

Lemma 17 **(1)** Let $P \approx_L Q$, in an AZDFS. Then $SFAM(P) = SFAM(Q)$ (or equivalently, $ST(P) \approx_{STA} ST(Q)$).
(2) If both P and Q are finite, then the converse is also true.

Proof

(1) Let $Fam(P'u) \in SFAM(P)$, where $P' + u \leq P$. Then u is Q/P'-needed, thus it has a residual Q^*v^* contracted in Q/P, implying that there is a redex $Q'v$ contracted in Q such that $v^* = v/(P'/Q')$. By Lemma 11, v is Q-needed, implying that $Fam(Q'v) = Fam(P'u) \in SFAM(Q)$. Thus $SFAM(P) = SFAM(Q)$. The rest of (1) is immediate by Lemmas 14 and 16 (since $ST(P)$ and $ST(Q)$ are standard by Theorem 12).
(2) By Lemma 16, $FAM(ST(P)) = FAM(ST(Q))$, implying $ST(P) \approx_{STA} ST(Q)$ by Theorem 12 and Lemma 14. But $P \approx_L ST(P)$ and $Q \approx_L ST(Q)$ by Theorem 12.(2), and (2) follows.

The following proposition summarizes our characterization of $\approx_L$ via zig-zag families in Lemmas 14, 16 and 17:

Proposition 18 Let P and Q be co-initial reductions in an AZDFS. Then

(1) If $P, Q \in STA$, then $P \approx_L Q$ iff $FAM(P) = FAM(Q)$.
(2) $SFAM(Q) = FAM(ST(Q)) = SFAM(ST(Q))$.
(3) If $P \approx_L Q$, then $SFAM(P) = SFAM(Q)$ (or equivalently, $ST(P) \approx_{STA} ST(Q)$). If both P and Q are finite, then the converse is also true.

3.3 The Projection Concept

Now we define the projection concept and show it is invariant under $\approx_L$.

Definition 19 Let P and Q be co-initial reductions in an AZDFS. Then we call $ST_P(ST(Q))$ the *projection* of Q onto P, written $Q|P$.

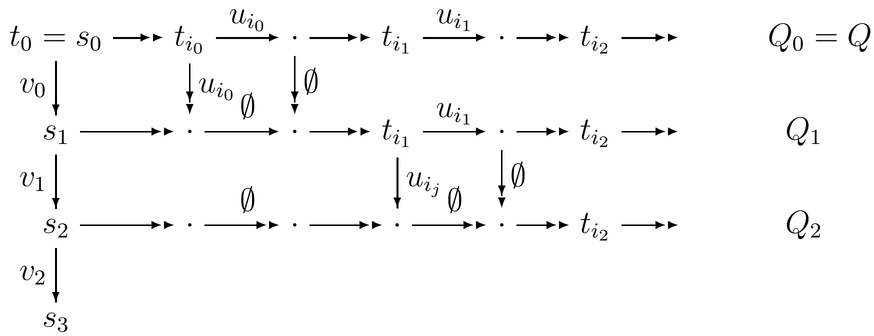
Lemma 20 Let $P, Q : t \twoheadrightarrow$, in an AZDFS. Then the first residual of a P-needed redex in t that is contracted in Q (if any) is the first step of Q in $SFAM(P)$.

Proof Let $Q : t \xrightarrow{u_0} t_1 \xrightarrow{u_1} \ldots$, let u_i be the first contracted residual of a P-needed redex in t, say of $v \subseteq t$, and let u_j be the first step of Q in $SFAM(P)$. Since v is P-needed, its P-needed residual is contracted in P by Lemma 11, hence $Fam(u_i) \in SFAM(P)$. Therefore, $j \leq i$. If u_j is a residual of a redex $v' \subseteq t$, then $Fam(v') \in SFAM(P)$, and by Proposition 7 the step of P in the family of v' must be a residual of v', thus v' is P-needed by Lemma 11; therefore in this case $i \leq j$ by the minimality of i, hence $i = j$. Otherwise, if (an ancestor of) u_j is created by u_k ($k < j$), then by [creation] $Fam(u_k) \hookrightarrow Fam(u_j)$; thus $Fam(u_k) \in SFAM(P)$ (since $SFAM(P) = FAM(ST(P))$ by Proposition 18.(2), and is downwards closed, w.r.t. $\hookrightarrow$, by [contribution]), contradicting the minimality of j. Hence $i = j$ and the lemma follows.

Lemma 21 Let P, Q be co-initial (not necessarily finite) reductions in an AZDFS. Then $FAM(ST_P(Q)) = SFAM(ST_P(Q)) = FAM(Q) \cap SFAM(P)$;

Proof Let $Q : t_0 \xrightarrow{u_0} t_1 \xrightarrow{u_1} \to \ldots$, let $u_{i_0}, u_{i_1}, \ldots$ be all steps of Q in $SFAM(P)$ ($i_0 < i_1 < \ldots$), and let $ST_P(Q) = v_0 + v_1 + \ldots$. Then by Lemma 15 and

Lemma 20 we have the following situation:



where u_{i_j} is a residual of v_j along $Q_j = Q/(v_0 + \ldots + v_{j-1})$, and $u_{i_j}, u_{i_{j+1}}, \ldots$ are all steps of Q_j in $SFAM(P_j)$, where $P_j = P/(v_0 + \ldots + v_{j-1})$. Hence $FAM(ST_P(Q)) = FAM(Q) \cap SFAM(P)$ and it remains to apply Proposition 18.(2).

Note in the above lemma that if Q is standard, then $FAM(ST_P(Q)) = SFAM(Q) \cap SFAM(P)$. The same holds if we require $Q \trianglelefteq_L P$ instead of $Q \in STA$; this can easily be shown using Lemma 10 and Lemma 11. The next example shows that the requirement that $Q \trianglelefteq_L P$ or $Q \in STA$ is necessary for the above equation to hold.

Example 22 Let $P = u$ and $Q = u+v'$, where $v' = v/u$ for some v and $u/v = \emptyset$. Then u is Q-unneeded, Q is not standard, and $Q \not\trianglelefteq_L P$. Further, $ST_P(Q) = u$, $FAM(ST_P(Q)) = FAM(Q) \cap SFAM(P) = Fam(u)$, but $SFAM(Q) \cap SFAM(P) = Fam(uv') \cap Fam(u) = \emptyset$.

Corollary 23 Let P, Q be co-initial reductions in an AZDFS. Then $P|Q$ is standard, and $FAM(P|Q) = SFAM(P|Q) = SFAM(Q) \cap SFAM(P)$. Hence $P|Q \approx_{STA} Q|P$.

Lemma 24 Let $P \approx_L P'$ and $Q \approx_{STA} Q'$, in an AZDFS. Then $ST_P(Q) \approx_{STA} ST_{P'}(Q')$.

Proof We have $SFAM(ST_P(Q)) = SFAM(Q) \cap SFAM(P) = SFAM(Q') \cap SFAM(P') = SFAM(ST_{P'}(Q'))$ by Lemma 21 and Proposition 18, implying by Proposition 18 and Theorem 12 that $ST_P(Q) \approx_L ST_{P'}(Q')$.

Corollary 25 The projection concept is invariant under $\approx_L$. (That is, if P, Q are co-initial reductions in an AZDFS, such that $P \approx_L P'$ and $Q \approx_L Q'$, then $P|Q \approx_L P'|Q'$.)

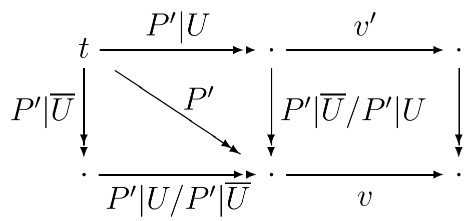
Note that, if we had defined $Q|P$ as $ST_P(Q)$ rather than $ST_P(ST(Q))$, the above corollary would fail: just take P and Q as in Example 22, and take $Q' = v$. Then $Q \approx_L Q'$, $Q|P = u$, but $Q'|P = \emptyset$.

We start with a simple definition.

Definition 26 Let $P : t \twoheadrightarrow$ be a reduction in a DRS, and let $U \subseteq t$ be a set of redexes in t. We call P *internal* to U or a *U-reduction* if it is external to $\overline{U}$, that is, if it contracts only residuals of redexes in U and created redexes. We call such redexes *U-redexes.*

Definition 27 (**Restrictions of reductions to redex-sets**) Let t be a term in an AZDFS $\mathcal{F}$, let $U \subseteq t$, and let $P : t \twoheadrightarrow$.

(1) Assume first that P is finite and standard. The concepts P *respects* U and the *restriction of P to U*, written $P|U$, are defined by induction on $n = |P|$ as follows. If $n = 0$, then P respects U and $P|U = \emptyset$. Now let $P = P' + v$ and let P' respect U. Assume that $P'|U$ and $P'|\overline{U}$ are defined as reductions internal to U and $\overline{U}$, respectively, such that $P' \approx_{STA} P'|U \sqcup P'|\overline{U}$. Then we say that P respects U if either $v = v'/(P'|\overline{U}/P'|U)$ for $v' \subseteq ft(P'|U)$ such that $(P'|U)+v'$ is still internal to U, or $v = v'/(P'|U/P'|\overline{U})$ for $v' \subseteq ft(P'|\overline{U})$ such that $(P'|\overline{U}) + v'$ is still internal to $\overline{U}$. In the first case (depicted on the picture below), we define $P|U = P'|U + v'$ and $P|\overline{U} = P'|\overline{U}$, and define $P|U = P'|U$ and $P|\overline{U} = P'|\overline{U} + v'$ in the second case.



(2) Now let P be infinite and standard. Then P respects U iff every $P^* \leq_{fin} P$ does, and $P|U$ is the reduction whose finite initial parts are of the form $P^*|U$, for all $P^* \leq_{fin} P$.

(3) We say that a (finite or infinite) reduction Q respects U if so does $ST(Q)$, and define $Q|U = ST(Q)|U$. We say that Q *respects* $\Im = \{U_i\}_{i \in I}$ if it respects every U_i.

The intuition is that, Q respects U iff $ST(Q)$ contracts only redexes to which only redexes in U contribute, or only those in $\overline{U}$, but not redexes in both U and $\overline{U}$. This intuition will be made precise in Proposition 30 below. Clearly, Q-respects U iff it respects $\overline{U}$.

In the above definition, we need to take a standard variant of Q before restricting it to U to ensure that the restriction notion is invariant under Lévy-equivalence. The following simple example shows why this step is necessary: Let $R = \{f(x) \to a,\ g(x) \to x\}$, let $Q : f(g(x)) \xrightarrow{v} f(x) \xrightarrow{u} a$, and let $U = \{v\}$.

Then 'direct restriction' of Q to U is v, while $Q|U = ST(Q)|U = \emptyset$, and $v \not\approx_L \emptyset$.

Lemma 28 **(1)** Definition 27 is correct, that is, for any finite and standard P in an AZDFS, $P \approx_{STA} P|U \sqcup P|\overline{U}$.
(2) Furthermore, $ST(P) \approx_{STA} P|U \sqcup P|\overline{U}$ is true for any (finite or infinite) P as well. (In particular, $P|U$ is P-needed and standard.)

Proof

(1) Assume first that P is finite and standard. Suppose that $P = P' + v$ respects U and that $v = v'/(P'|\overline{U}/P'|U)$ for $v' \subseteq ft(P'|U)$ such that $(P'|U)+v'$ is internal to U. Then $P = P'+v \approx_L (P'|U \sqcup P'|\overline{U})+v = P'|U + P'|\overline{U}/(P'|U) + v \approx_L P'|U + v' + P'|\overline{U}/(P'|U + v') = P|U + P|\overline{U}/(P|U) = P|U \sqcup P|\overline{U}$. The case when $v = v'/(P'|U/P'|\overline{U})$ for $v' \subseteq ft(P'|\overline{U})$ such that $(P'|\overline{U})+v'$ is internal to $\overline{U}$ is similar. Finally, $P|U \sqcup P|\overline{U}$ is P-needed, hence standard, by Lemma 11, and (1) follows.
(2) If P is finite, then (2) follows immediately from (1) (since $ST(P)$ is standard by Theorem 12, and $P|U = ST(P)|U$ and $P|\overline{U} = ST(P)|\overline{U}$ by Definition 27.(3)). So let P be infinite. Since, for any $P^* \leq_{fin} P$, $ST(P^*) \approx_{STA} P^*|U \sqcup P^*|\overline{U} \trianglelefteq_L ST(P)|U \sqcup ST(P)|\overline{U}$, we have immediately, by definition of $\trianglelefteq_L$, that $ST(P) \trianglelefteq_L ST(P)|U \sqcup ST(P)|\overline{U}$. For the converse, let $P' \leq_{fin} ST(P)|U \sqcup ST(P)|\overline{U}$. Then there is a reduction $P'^* \leq_{fin} P$ such that $P' \trianglelefteq_L ST(P'^*)|U \sqcup ST(P'^*)|\overline{U} \approx_L ST(P'^*) \trianglelefteq_L ST(P)$. Hence $ST(P)|U \sqcup ST(P)|\overline{U} \trianglelefteq_L ST(P)$. Thus $ST(P)|U \sqcup ST(P)|\overline{U} \approx_L ST(P) = P|U \sqcup P|\overline{U}$. But $ST(P)|U \sqcup ST(P)|\overline{U}$ is $ST(P)$-needed, hence standard, by Lemma 11, and the lemma follows.

As in the case of the projection concept (see Corollary 23), we need to give a characterization of the restriction concept via families in order to allow for simpler proofs in the sequel.

Notation 29 *Below, for any $U \subseteq t$ in a DFS, $FAM_0(U)$ will denote the set of families (relative to t) of redexes in U, and $FAM_0^+(U)$ will denote the minimal set of families containing $FAM_0(U)$ and closed under the contribution relation.*

Proposition 30 Let $U \subseteq t$ and $P : t \twoheadrightarrow$, in an AZDFS. Then P respects U iff $SFAM(P) \subseteq FAM_0^+(U) \cup FAM_0^+(\overline{U})$. In the latter case, $SFAM(P|U) = FAM(P|U) = SFAM(P) \cap FAM_0^+(U)$ and $SFAM(P) = FAM(P|U) \cup FAM(P|\overline{U})$.

Proof By Proposition 18 and Definition 27, we can assume that $P \in STA$. Furthermore, we can assume that P is finite. Then the first part is proved by induction on $|P|$ by simply using Definition 27. $FAM(P|U) = FAM(P) \cap FAM_0^+(U)$ and $FAM(P) = FAM(P|U) \cup FAM(P|\overline{U})$ also follow immediately from Definition 27 by induction on $|P|$, since $P, P|U, P|\overline{U} \in STA$ by

Lemma 28.

Lemma 31 If P respects U and $Q \approx_L P$, in an AZDFS, then so does Q, and $P|U \approx_{STA} Q|U$.

Proof By Proposition 30 and Proposition 18 (since $P|U, Q|U$ are standard by Lemma 28).

The restriction concept enjoys nice algebraic properties:

Lemma 32 Let $P : t \twoheadrightarrow$ respect $U_1, U_2 \subseteq t$, in an AZDFS. Then

(1) P respects $U_1 \cup U_2$ and $P|U_1 \cup U_2 \approx_{STA} P|U_1 \sqcup P|U_2$;
(2) P respects $U_1 \cap U_2$, $P|U_1$ respects U_2, and $P|U_1 \cap U_2 \approx_{STA} (P|U_1)|U_2$.
(3) P respects $U_1 \setminus U_2$, $P|U_1$ respects $\overline{U_2}$, and $P|U_1 \setminus U_2 \approx_{STA} (P|U_1)|\overline{U_2}$.

Proof By Definition 27, we can assume that P is standard and finite.

(1) Since P respects U_i, we have by Proposition 30 that $\forall \phi \in FAM(P) : \phi \in FAM_0^+(U_i) \vee \phi \in FAM_0^+(\overline{U_i})$. In each of the four cases, $\phi \in FAM^+(U_1 \cup U_2)$ or $\phi \in FAM^+(\overline{U_1 \cup U_2})$, implying by Proposition 30 that P respects $U_1 \cup U_2$. (For example, if $\phi \in FAM^+(U_1)$ and $\phi \in FAM^+(\overline{U_2})$, then $\phi \in FAM^+(U_1 \setminus U_2)$, hence $\phi \in FAM^+(U_1 \cup U_2)$.) Since $P|U_i \trianglelefteq_L P$ and are P-needed, $P|U_1 \sqcup P|U_2 \trianglelefteq_L P$ and is P-needed too, and therefore standard, by Lemma 10 and Lemma 11. Thus $FAM(P|U_1) \cup FAM(P|U_2) = FAM(P|U_1 \sqcup P|U_2)$. If there was a family $\psi \in FAM(P) \cap (FAM_0^+(U_1 \cup U_2) \setminus (FAM_0^+(U_1) \cup FAM_0^+(U_2)))$, then there would be $\psi_1, \psi_2 \hookrightarrow_z \psi$ such that $\psi_1 \in U_1$ and $\psi_2 \in U_2 \setminus U_1 \subseteq \overline{U_1}$, which is impossible by Proposition 30 since P respects U_1. So $FAM(P|U_1 \cup U_2) = FAM(P|U_1) \cup FAM(P|U_2) = FAM(P|U_1 \sqcup P|U_2)$, implying by Proposition 18 that $P|U_1 \cup U_2 \approx_{STA} P|U_1 \sqcup P|U_2$ (since $P|U_1 \cup U_2)$ is standard by Lemma 28).
(2) Clearly, $\overline{U_1 \cap U_2} = \overline{U_1} \cup \overline{U_2}$, hence P respects $U_1 \cap U_2$ (and $\overline{U_1 \cap U_2}$) by (1). Further, since $P \approx_{STA} P|(U_1 \cup \overline{U_1}) \approx_{STA} P|U_1 \sqcup P|\overline{U_1}$ by (1) and P respects U_2, $P|U_1$ also respects U_2 by Lemma 31. Finally, we have by Proposition 30 that $SFAM(P|U_1 \cap U_2) = SFAM(P) \cap FAM^+(U_1 \cap U_2) = SFAM(P) \cap FAM^+(U_1) \cap FAM^+(U_2) = SFAM(P|U_1) \cap FAM^+(U_2) = SFAM((P|U_1)|U_2)$, implying $P|U_1 \cap U_2 \approx_{STA} (P|U_1)|U_2$ by Proposition 18 and Lemma 28.
(3) From (2), since $U_1 \setminus U_2 = U_1 \cap \overline{U_2}$.

3.5 The Decomposition Theorem

The following lemma relates the restriction and projection concepts:

Lemma 33 Let $U \subseteq t$, in an AZDFS , and let P and $\overline{P}$ be internal to U and $\overline{U}$, respectively. Then $P \sqcup \overline{P}$ respects U and $(P \sqcup \overline{P})|U \approx_{STA} (P \sqcup \overline{P})|P \approx_{STA} P|(P \sqcup \overline{P})$.

Proof We have by Proposition 30 and Corollary 23 that $FAM((P \sqcup \overline{P})|U) = SFAM(P \sqcup \overline{P}) \cap FAM^{+}(U) =$ (by Definition 9, since $\overline{P}$ is external to U) $FAM(P) \cap SFAM(P \sqcup \overline{P}) =$ (since $P \sqcup \overline{P}$-needed redexes contracted in P are P-needed by Lemma 10) $= SFAM(P) \cap SFAM(P \sqcup \overline{P}) = SFAM(P|P \sqcup \overline{P})$. Now the lemma follows from Proposition 18 and Corollary 23.

Definition 34 Let U_i with $i \in I$ be nonempty sets of redexes in t such that $\cup_{i \in I} U_i$ contains each redex of t and $U_i \cap U_j = \emptyset$ when $i \neq j$. Then we call the set $\Im = \{U_i\}_{i \in I}$ a *(redex-)covering* of t.

We can now prove the main result of this section.

Theorem 35 Let $\Im = \{U_i\}_{i \in I}$ be a redex-covering of a term t in an AZDFS $\mathcal{F}$.

(1) Let P_i be internal to U_i, and let $P = \sqcup_i P_i$. Then P respects $\Im$ and $P|U_i \approx_{STA} P|P_i$.
(2) Let $P : t \twoheadrightarrow$ respect $\Im$. Then $ST(P) \approx_{STA} \sqcup_i P|U_i$. If moreover P is finite, then $P \approx_L \sqcup_i P|U_i$

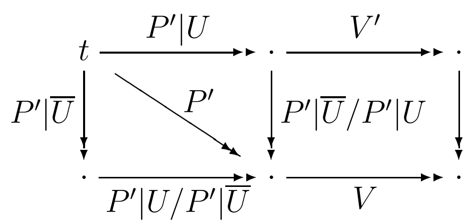
Proof

(1) Let $\overline{P_i} = \sqcup_{j \neq i} P_j$ $(i \in I)$. Then $P \approx_L P_i \sqcup \overline{P_i}$ and $\overline{P_i}$ is internal to $\overline{U_i}$. Hence, by Lemma 33 and Lemma 31, P respects U_i, thus P respects $\Im$, and $P|U_i \approx_{STA} P|P_i$.
(2) Let $I = \{1, \ldots, n\}$. Then, since P respects $\Im$, we have by Lemma 32.(1) that $ST(P) \approx_{STA} P|(U_1 \cup \ldots \cup U_n) \approx_{STA} P|U_1 \sqcup P|(U_2 \cup \ldots \cup U_n) \approx_{STA} \ldots \approx_{STA} P|U_1 \sqcup \ldots \sqcup P|U_n$. If moreover P is finite, then we apply Theorem 12.(2).

Now Remark 6 allows us to derive a counterpart of Theorem 35 for DFSs in general. Let us take a closer look say at the definition of the restriction concept, and spell out explicitly the corresponding definition for DFSs.

Definition 36 (**Restrictions of complete family-reductions to redex-sets**) Let t be a term in a DFS $\mathcal{F}$, let $U \subseteq t$, and let $P : t \twoheadrightarrow$ be a complete family-reduction

(1) Assume first that P is finite and standard. The concepts P *respects* U and the *restriction of* P *to* U, written $P|U$, are defined by induction on the number n of multi-steps in P as follows. If $n = 0$, then P respects U and $P|U = \emptyset$. Now let $P = P' + V$ and let P' respect U. Assume that $P'|U$ and $P'|\overline{U}$ are defined as complete family-reductions internal to U and $\overline{U}$, respectively, such that $P' \approx_{STA} P'|U \sqcup P'|\overline{U}$. Then we say that P respects U if either $V = V'/(P'|\overline{U}/P'|U)$ for $V' \subseteq ft(P'|U)$ such that $(P'|U) + V'$ is still internal to U, or $V = V'/(P'|U/P'|\overline{U})$ for $V' \subseteq ft(P'|\overline{U})$ such that $(P'|\overline{U}) + V'$ is still internal to $\overline{U}$, where V' is the maximal set of redexes in $ft(P'|U)$ or $ft(P'|\overline{U})$, respectively, that belong to the family of $P'v$ $(v \in V)$. In the first case (depicted on the picture below), we define $P|U = P'|U + V'$ and $P|\overline{U} = P'|\overline{U}$, and define $P|U = P'|U$ and $P|\overline{U} = P'|\overline{U} + V'$ in the second case.



(2) Now let P be infinite and standard. Then P respects U iff every complete family-reduction $P^* \leq_{fin} P$ does, and $P|U$ is the complete family-reduction reduction whose finite initial parts are of the form $P^*|U$, for all $P^* \leq_{fin} P$.
(3) We say that a complete family-reduction Q respects U if so does $ST(Q)$, and define $Q|U = ST(Q)|U$. We say that Q *respects* $\Im = \{U_i\}_{i \in I}$ if it respects every U_i.

In the above definition, a complete family-reduction P is standard if every multi-step in P is P-needed, i.e., contracts at least one P-needed redex. When standardizing complete family-reductions, entire multi-steps are swapped. It is shown in [19,20] that in any DFS: (a) if $U, V \subseteq s$ are complete sets of redexes of families ϕ and ψ in s, respectively, and $s \overset{V}{\twoheadrightarrow} o$, then $U' = U/V$ is the complete set of redexes of ϕ in o. This ensures that swapping of multi-steps can be done 'safely'. (Clearly, a sequentialization of a standard multi-step reduction need not be a standard reduction.) The meanings of $\leq_{fin}$, $ST(\ )$ and $\approx_{STA}$ for multi-step reductions change respectively. Again from property (a) above we know that, in the above definition, it is correct to take for V' the set of all redexes in $ft(P'|U)$ (respectively, $ft(P'|\overline{U})$) in the family of V as such a V' is a (complete) family multi-step.

Thus, we can equivalently define the relative standardization, projection, and restriction concepts for complete family reductions in a DFSs $\mathcal{F}$ by first mapping the corresponding complete family-reductions onto reductions in the corresponding AZDFS $\mathcal{F}_I$, then performing the corresponding operations on these reductions, in $\mathcal{F}_I$, and finally mapping the result back onto the corresponding complete family-reduction in $\mathcal{F}$. Hence the following is a corollary of Theorem 35.

Corollary 37 (Decomposition Theorem) Let $\Im = \{U_i\}_{i \in I}$ be a redex-covering of a term t in a DFS $\mathcal{F}$.

(1) Let P_i be complete family-reductions internal to U_i, and let $P = \sqcup_i P_i$. Then $P, P|U_i$ and $P|P_i$ are complete family-reductions such that P respects $\Im$ and $P|U_i \approx_{STA} P|P_i$.
(2) Let $P : t \twoheadrightarrow$ be a complete family-reduction that respects $\Im$. Then $ST(P), P|U_i$ are complete family-reductions such that $ST(P) \approx_{STA} \sqcup_i P|U_i$. If moreover P is finite, then $P \approx_L \sqcup_i P|U_i$.

3.6 The Decomposition Theorem in DEESs

We now recall an event model, *Deterministic Erasure Event Structures* (DEESs), which is equivalent to AZDFSs [18,21]. This equivalence allows us to freely choose between the two models as a formalism for developing our intermediate decomposition results. Since our goal is to study the decomposition properties for duplicating systems, and the restriction to AZDFSs is only a technical tool simplifying the proofs, we have demonstrated above our constructions on the operational (semantic) level (AZDFSs). Now we will interpret our constructions on the event (semantic) level (DEESs), to support our claim of *distributedness* or *parallelism* in computation offered by our independence concept.

Recall that a *Deterministic (or Conflict-free) Prime Event Structure* (DPES) [37] is a couple $\mathcal{E} = (E, \leq)$, where E is a set of *events*, ranged over by $e, e_1, \ldots$, and the *causal dependency relation* $\leq$ is a partial order on E, such that the set $\lceil e^< \rceil = \{e' \,|\, e' < e\}$ is finite for every $e \in E$. Finite *configurations* of $\mathcal{E}$ are finite *left-closed subsets* $\alpha, \beta, \ldots$ of E, i.e., subsets $\mathcal{L}_{fin}(E) = \{\alpha \subseteq_{fin} E \,|\, e \in \alpha \wedge e' < e \Rightarrow e' \in \alpha\}$; they represent stages of computation.

Definition 38 A *Deterministic Erasure Event Structure* (DEES) is a triple $\mathcal{C} = (E, \leq, \rhd)$, where $\mathcal{E} = (E, \leq)$ is a DPES and $\rhd \subseteq \mathcal{L}_{fin}(E) \times E$ is the *inessentiality* or *erasure* relation (read $\alpha \rhd e$ as: 'e is α-*inessential*'), satisfying the following axioms, where $\alpha, \beta \in \mathcal{L}_{fin}(E)$:

[E0] $\forall e \in E : \emptyset \not\rhd e$;
[E1] $\alpha \rhd e \wedge \alpha \subseteq \beta \in \mathcal{L}_{fin}(E) \Rightarrow \beta \rhd e$;
[E2] $\alpha \rhd e' \wedge \alpha \rhd e \wedge \alpha - e' \in \mathcal{L}_{fin}(E) \Rightarrow \alpha - e' \rhd e$;
[E3] $\alpha \rhd e \wedge e < e' \;\Rightarrow\; \alpha \rhd e'$;
[E4] $\alpha \cup \lceil e^< \rceil \rhd e \;\Rightarrow\; \alpha \rhd e$.

The erasure relation is extended to infinite configurations α by: $\alpha \rhd e$ iff $\exists \alpha' \subseteq_{fin} \alpha : \alpha' \rhd e$.

We will call events that are smallest w.r.t. the causal dependency relation, $\leq$, *initial*. The set of all initial events in E will be denoted by $Init(E)$, and similarly for any subset of events (e.g., $Init(\alpha)$). We will use X, Y to range over sets of initial events. We will assume that $Init(E)$ is finite.

The (isomorphic) translation of an AZDFSs $\mathcal{F}$ into the corresponding DEESs $\mathcal{C}$ [18,21] interprets a reduction P in $\mathcal{F}$ into the configuration $FAM(P)$ of $\mathcal{C}$ (families in $\mathcal{F}$ are events in $\mathcal{C}$). Hence, by Corollary 23, the projection concept for configurations in DEESs can be defined as follows: $\alpha|\beta = ST(\alpha) \cap ST(\beta)$, where $ST(\alpha)$, the *standard variant* of α, is defined by $ST(\alpha) = \{e \in \alpha \mid \alpha \not\rhd e\}$.

Further, let α be a configuration in a DEES $\mathcal{C}$ and let X be a set of *initial* events in $\mathcal{C}$. By Proposition 30, we can define that α *respects* X if $ST(\alpha) \subseteq \lceil X^{\geq} \rceil \cup \lceil \overline{X}^{\geq} \rceil$, and in the latter case, define $\alpha|X = ST(\alpha) \cap \lceil X^{\geq} \rceil$, where $\lceil X^{\geq} \rceil = \{e \mid \exists e' \in X : e \geq e'\}$.

Now the Decomposition Theorem can be translated in DEESs as follows:

Theorem 39 (Decomposition Theorem) Let $\Im = \{X_i\}_{i \in I}$ be a covering of the set of initial events in a DEES $\mathcal{C}$. Further:

(1) Let $\alpha_i \subseteq \lceil X_i^{\geq} \rceil$, $i \in I$, and let $\alpha = \cup_{i \in I} \alpha_i$. Then α respects (all sets in) $\Im$ and $\alpha|X_i = \alpha|\alpha_i$.
(2) Let α respect $\Im$. Then $ST(\alpha) = \cup_{i \in I}(\alpha|X_i)$.

4 The Geometry of Conflict-free Reduction Spaces

In this section, we introduce the *Reduction Geometry* and prove the *Independent Decomposition Theorem*, which reflects the main analogy of orthogonal reduction spaces with the Euclidean Geometry.

Definition 40 Let $P : t \twoheadrightarrow$. We call the *strict domain* of P, $SDom(P)$, the minimal set of redexes $U \subseteq t$ such that P is internal to U. We call the *domain* of P, written $Dom(P)$, the set $\cup_{Q \approx_L P} SDom(Q)$, i.e., the minimal set of redexes $U \subseteq t$ such that any Q that is Lévy-equivalent to P is internal to U. We call the *minimal domain* of P, written $MDom(P)$, the set $\cap_{Q \approx_L P} SDom(Q)$.

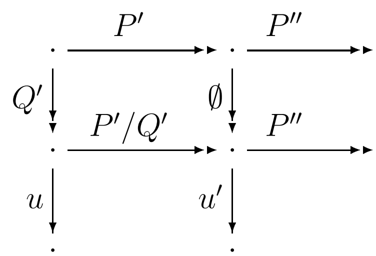
It is easy to see that $Dom(P)$ is $SDom(P)$ augmented with all P-erased redexes not contracted in P; $MDom(P)$ is the set of all P-needed redexes in t; and $Dom(P) \setminus MDom(P)$ is the set of all P-unneeded P-erased redexes in t. Obviously $P \approx_L Q$ implies $Dom(P) = Dom(Q)$ and $MDom(P) = MDom(Q)$, but not $SDom(P) = SDom(Q)$. It follows from Theorem 12 (and also from Proposition 18 in the finite case) that $MDom(P) = SDom(ST(P))$ for any P.

Definition 41 A U-reduction P is called *U-fair* if each U-redex in a term in P is erased in P, and P is called *strongly U-cofinal* if, for any U-reduction Q, $Q \trianglelefteq_L P$. If U is the set of all redexes in t, then U-fair reductions are called *fair*, and strongly U-cofinal reductions will be called *strongly cofinal*.

Lemma 42 A U-reduction P, in an AZDFS , is U-fair iff it is strongly U-cofinal.

Proof

($\Rightarrow$) Let Q be a U-reduction. We want to show that $Q \trianglelefteq_L P$. Suppose it is not. It follows from the definition of $Q \trianglelefteq_L P$, for possibly infinite reductions P and Q, that Q has the form $Q = Q' + u + Q''$, where $Q'/P = \emptyset$ and $u/(P/Q') \neq \emptyset$. Let $P = P' + P''$, where P' is finite and $Q/P' = \emptyset$ (such a P' must exist by definition of $\trianglelefteq_L$), and let $u' = u/(P'/Q')$ (see the figure). Then by U-fairness of P, $u'/P'' = \emptyset$ (note that u' is a redex in a term in P), contradicting $u/(P/Q') \neq \emptyset$.



($\Leftarrow$) Suppose on the contrary that P is not U-fair. Then $P = P' + P''$ where P' is finite and $ft(P')$ contains a U-redex u that is not erased in P''. Clearly $P^* = P' + u \sqcup P''$ is also a U-reduction and $P^* \not\trianglelefteq_L P$ – a contradiction.

The following lemma will not be used below, but it helps in understanding the relationship between the projection and restriction concepts: projection of Q onto P has an effect similar to restriction of Q to the set of P-needed redexes. See [22] for the proof.

Lemma 43 Let Q and P be co-initial reductions in an AZDFS $\mathcal{F}$. Then $Q|P \approx_{STA} (Q|MDom(P))|P$. If moreover $SFAM(P) = FAM_0^+(MDom(P))$ (e.g., if $\mathcal{F}$ is linear and P is $MDom(P)$-fair), then $Q|P \approx_{STA} Q|MDom(P)$.

Definition 44 (The Reduction Geometry) Let an AZDFS $\mathcal{F}$ be given.

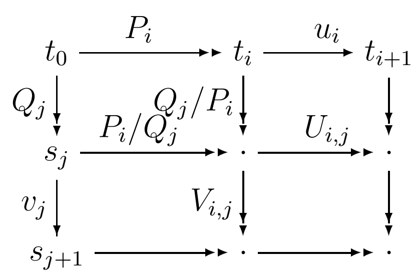
- Two co-initial reductions $P : t \twoheadrightarrow s$ and $Q : t \twoheadrightarrow e$ are said *not to interact*, written $P \perp Q$, if $MDom(P) \cap MDom(Q) = \emptyset$, and any created redex in $ft(P \sqcup Q)$ is a residual of a redex either from $ft(P)$ or from $ft(Q)$. Otherwise, we say that P and Q (where $MDom(P) \cap MDom(Q) = \emptyset$) *interact*, written $P \not\perp Q$.
- We call a set $\Pi = \{P_i\}_{i \in I}$ of reductions starting from t *independent* if $P'_i \perp \sqcup_{i \neq j} P'_j$ for every $i \in I$ and any $P'_i \leq_{fin} P_i$. We call Π a *basis* of $\mathcal{F}$ at t if Π is independent and $\sqcup P_i$ is maximal w.r.t. $\trianglelefteq_L$.
- The *distance* $\|P, Q\|$ between co-initial reductions $P, Q : t \twoheadrightarrow$ is the number of families whose *essential* member redexes are contracted either in P or in Q (but not in both). Here a redex $v \subseteq s$ is essential [13,14] (or equiv-

alently, Maranget-needed [31]) if in any fair reduction starting from s a residual of v is contracted.

- The *independence degree* of $U \subseteq t$ is the length of a shortest finite P internal to U such that there exists a reduction Q external to U that interacts with P, and is ∞ otherwise.
- We call $U \subseteq t$ *independent* if no (finite) U-reduction interacts with a $\overline{U}$-reduction. We call $\Im = \{U_i\}_{i \in I}$ an *independent covering* if $\Im$ is a covering of t and each U_i is independent.

Example 45 (Bases) Consider a term t containing three redexes u, v, w. Let $w/(u \sqcup v) = \emptyset$, $w/u \neq \emptyset$, $w/v \neq \emptyset$, and assume no redex can be created by contraction of these redexes. Then $\Pi_1 = \{u, v\}, \Pi_2 = \{u, v, w\}, \Pi_3 = \{u, w \sqcup v\}$ and $\Pi_4 = \{u, v \sqcup w\}$ are all bases at t (there are others too), as all Π_i are independent, and $u \sqcup v \approx_L u \sqcup v \sqcup w \approx_L u \sqcup (w \sqcup v) \approx_L u \sqcup (v \sqcup w)$ are all normalizing, hence strongly cofinal. For Π_1, the strict domains of the axes do not form a covering of t, while for other bases they do. Note also that for Π_4, u erases the second step of $v \sqcup w$ – $(w/v)/(u/v) = \emptyset$.

Definition 46 (Externality) Let $P : t_0 \overset{P_i}{\twoheadrightarrow} t_i \overset{u_i}{\rightarrow} t_{i+1} \twoheadrightarrow t_n$ and $Q : t_0 = s_0 \overset{Q_j}{\twoheadrightarrow} s_j \overset{v_j}{\rightarrow} s_{j+1} \twoheadrightarrow s_m$. Let $U_{i,j} = u_i/(Q_j/P_i)$ and $V_{i,j} = v_j/(P_i/Q_j)$ (see diagram). We call P *external* to Q if for any i, j, $U_{i,j} \cap V_{i,j} = \emptyset$.



A reduction external to one complete development of U need not be external to all developments of U, and in general, externality is not invariant under $\approx_L$. For, consider a TRS $R = \{a \to a',\ f(x) \to b,\ g(x) \to c\}$, a term $t = f(g(a))$, and reductions $P : t \overset{a}{\rightarrow} f(g(a')) \overset{f}{\rightarrow} b$, $Q : t \overset{a}{\rightarrow} f(g(a')) \overset{g}{\rightarrow} f(c)$, and $N : t \overset{g}{\rightarrow} f(c)$. Then we have $Q \approx_L N$, P is external to N, but not to Q; and P is not external to $U = \{a,\ g(a)\}$.

Note that, in the definition of $P \perp Q$, a created redex in $ft(P \sqcup Q)$ cannot be a residual of redexes from both $ft(P)$ and $ft(Q)$ as otherwise the same redex would be a residual of redexes from $ft(ST(P))$ and $ft(ST(Q))$ (the Cube Lemma implies that $\perp$ is invariant under Lévy-equivalence), which is impossible by the Stability Lemma [3] (since $MDom(P) \cap MDom(Q) = \emptyset$ implies that

[3] The Stability Lemma [6,20] states that if $P : t \twoheadrightarrow s$ is external to $Q : t \twoheadrightarrow e$, in an SDRS, and P creates redexes $W \subseteq s$, then the residuals $W/(Q/P)$ of redexes in

$ST(P)$ and $ST(Q)$ are external).

In general, $SDom(P) \cap SDom(Q) = \emptyset$ implies that P and Q are external, but not conversely. For example, let $t = u = (\lambda z.Kyz)v$, where $K = \lambda x.\lambda y.x$; then $P : t \xrightarrow{u} Kyv \xrightarrow{v} Kyo$ and $Q : t \xrightarrow{K} (\lambda z.y)v \xrightarrow{v} (\lambda z.y)o$ are external since v does not have $u \sqcup K$-residuals, while $SDom(P) \cap SDom(Q) = \{v\} \neq \emptyset$.

Note also that, if $P \perp Q$, $Dom(P) \cap Dom(Q) = \emptyset$ need not hold: Consider the modified example from [30], taking $t = (\lambda x.K_a(xu))K_b$ where $K_a = \lambda x.a$, $K_b = \lambda x.b$, and u is a redex, and consider the reductions

$$P : t \xrightarrow{t} K_a(K_b u) \xrightarrow{u} K_a(K_b o) \xrightarrow{K_b} K_a b$$

and

$$Q : t \xrightarrow{u} (\lambda x.K_a(xo))K_b \xrightarrow{K_a} (\lambda x.a)K_b,$$

where o is the contractum of u. Then $u \in Dom(P)$, $Dom(Q)$, but $u \notin MDom(P)$, $MDom(Q)$, since u is not needed either in P or in Q.

In the definition of distance between reductions P and Q, one might think that it would be more appropriate to consider $P \sqcup Q$-needed redexes only. The following example shows that the distance would not be a metric: take $t = Kx\omega$, $P : t \xrightarrow{\omega} t \xrightarrow{\omega} t \xrightarrow{\omega} t \xrightarrow{\omega} t$, $Q : t \xrightarrow{\omega} t$, and $N : t \xrightarrow{\omega} t \xrightarrow{\omega} t \xrightarrow{\omega} t \xrightarrow{\omega} t \xrightarrow{K} x$. Then $\|P, Q\| = 3$ and $\|P, N\| = \|N, Q\| = 1$. It is easy to check that our distance measure on finite co-initial reductions satisfies the triangle inequality. To make it a metric, we define for co-initial finite reductions P, Q, $P \approx_E Q$ iff $EFAM(P) = EFAM(Q)$, where $EFAM(P)$ denotes the set of families of essential redexes in P. Clearly, $\approx_E$ is an equivalence relation, and the (co-initial) reduction space quotiented w.r.t. it is a metric, as $|P, Q| = 0$ implies $P \approx_E Q$. Note that $\approx_L \subseteq \approx_E$, but not conversely.

The independence degree of $U \subseteq t$, if finite, characterizes the minimal amount of work that can be performed in U independently from the rest of the computation. Clearly the independence degree of an independent redex set is ∞.

Lemma 47 In an AZDFS , $U \subseteq t$ is independent iff any reduction $P : t \twoheadrightarrow$ respects it.

Proof By Definition 27, we can assume that P is standard and finite.

(⇒) Suppose on the contrary that not every reduction starting from t respects U, and let $P = P' + u$ be a shortest standard reduction that does not respect U. Since P' respects U, we have that $P' \approx_{STA} P'|U \sqcup P'|\overline{U}$ by Lemma 28. Since P does not respect U, u is not the residual of a redex either in $ft(P'|U)$

W are created by P/Q, and Q/P is external to W.

or in $ft(P'|\overline{U})$. But this means that $P'|U$ and $P'|\overline{U}$ interact (since by the Cube Lemma u is a created redex), i.e., U is not independent.

($\Leftarrow$) Let U not be independent. Then there are reductions N and Q, respectively internal and external to U, that interact, i.e., there is a new redex u in $ft(N \sqcup Q)$ that is not a residual of a redex either in $ft(N)$ or in $ft(Q)$. If $N \sqcup Q$ does not respect U, then we are done. Otherwise, say $(N \sqcup Q)|U \sqcup (N \sqcup Q)|\overline{U} + u$ does not respect U.

Proposition 48 Let $U_1, U_2 \subseteq t$ be independent, in an AZDFS. Then so are $U_1 \cup U_2$, $U_1 \setminus U_2$, and $U_1 \cap U_2$.

Proof Immediate from Lemma 32 and Lemma 47.

Theorem 49 Let $\Im = \{U_i\}_{i \in I}$ be an independent redex-covering of a term t in an AZDFS $\mathcal{F}$, let $P : t \twoheadrightarrow$, and let P_i be U_i-fair. Then $ST(P) \approx_{STA} \sqcup_i P|U_i$. Further, $B = \{P_i\}_{i \in I}$ is a basis at t, and there are U_i-reductions $P'_i \trianglelefteq_L P_i$ such that $ST(P) \approx_{STA} \sqcup_i P'_i$. If moreover P is finite, then $P \approx_L \sqcup_i P|U_i \approx_L \sqcup_i P'_i$.

Proof Since $\Im$ is an independent covering of t, P respects $\Im$ by Lemma 47, and $ST(P) \approx_{STA} \sqcup_i P|U_i$ follows from Theorem 35.(2). Since P_i is U_i-fair and $P|U_i$ is an U_i-reduction, we have $P|U_i \trianglelefteq_L P_i$ by Lemma 42. Hence $P \trianglelefteq_L \sqcup P_i$. It follows immediately from Definition 44 that B is independent, thus is a basis at t, and we can take $P|U_i$ for P'_i. If moreover P is finite, then we apply Theorem 12.(2).

By Remark 6, Definition 44 induces the concepts of a basis, an independent redex-covering, etc. for complete family-reductions in DFSs. Hence the following is an immediate corollary of Theorem 49.

Corollary 50 (Independent Decomposition Theorem) Let $\Im = \{U_i\}_{i \in I}$ be an independent redex-covering of a term t in a DFS $\mathcal{F}$, let $P : t \twoheadrightarrow$ be a complete family-reduction, and let P_i be U_i-fair complete family-reductions. Then $P|U_i$ are complete family-reductions such that $ST(P) \approx_{STA} \sqcup_i P|U_i$. Further, $B = \{P_i\}_{i \in I}$ is a basis at t, and there are complete family-reductions $P'_i \trianglelefteq_L P_i$ internal to U_i such that $ST(P) \approx_{STA} \sqcup_i P'_i$. If moreover P is finite, then $P \approx_L \sqcup_i P|U_i \approx_L \sqcup_i P'_i$.

We have seen in Example 45 that not all bases are of the form described in Theorem 49. That is, if $\{P_i\}_{i \in I}$ is a basis at t, P_i need not be an U_i-fair reduction for some independent covering $\Im = \{U_i\}_{i \in I}$ of t, as it is the case for Π_1 (since $w/u \neq \emptyset$ and $w/v \neq \emptyset$). We could exclude this situation, by requiring in the definition of independence of $U \subseteq t$ that for any pair of finite reductions P and Q, respectively internal and external to U, Q does not erase any steps of P, that is, $|P| = |P/Q|$. We have chosen not to do so in order to avoid a trivial concept of independence (see also the introduction). Also in the case of relativized bases which we introduce in the next section, axes do not need to be maximal reductions on their strict domains.

Note that every term t in an AZDFS has an independent redex covering – $\{U(t)\}$, where $U(t)$ is the set of all redexes of t, and has an independent basis – a fair reduction starting from t. Theorem 49 and Proposition 48 allow us to construct finer independent coverings and finer bases from existing ones, as if $\Im = \{U_i\}_{i\in I}$ and $\Im' = \{U'_j\}_{j\in J}$ are independent coverings, then $\Im \cap \Im' = \{U_i \cap U'_j\}_{(i,j)\in(I,J)}$ is a one too. Thus t has a unique finest independent covering (consisting of finest independent redex sets). [4]

We conclude this section by translating the concept of Reduction Geometry and Theorem 49 in the terminology of DEESs.

Definition 51 Let a DEES $\mathcal{C} = (E, \leq, \rhd)$ be given.

- Let α and β be two finite configurations such that $Init(ST(\alpha)) \cap Init(ST(\beta)) = \emptyset$. We say that α and β *do not interact*, written $\alpha \perp \beta$, if for any event e such that $\alpha \cup \beta \cup \{e\}$ is a configuration, at least one of $\alpha \cup \{e\}$, $\beta \cup \{e\}$ is a configuration too. Otherwise, α and β *interact*, written $\alpha \not\perp \beta$.
- We call a set $\Pi = \{\alpha_i\}_{i\in I}$ of configurations *independent* if $\alpha'_i \perp \cup_{j\neq i}\alpha'_j$ for any $i \in I$ and $\alpha'_i \subseteq_{fin} \alpha_i$. We call Π a *basis* for $\mathcal{C}$ if it is independent and $ST(\alpha) \subseteq \cup_{i\in I}\alpha_i$ for any configuration α.
- An event e is called *essential* if $\alpha \not\rhd e$ for any finite configuration α. The *distance* $\|\alpha, \beta\|$ between α and β is the number of essential events that belong to either α or β (but not to both).
- The *independence degree* of $X \subseteq Init(E)$ is the cardinality of a smallest configuration $\alpha \subseteq_{fin} \lceil X^{\geq} \rceil$ such that there exists a configuration $\beta \subseteq_{fin} \lceil \overline{X}^{\geq} \rceil$ that interacts with α, and is ∞ otherwise. (Here $\overline{X} = Init(E) \setminus X$.)
- We call $X \subseteq Init(E)$ *independent* if for any configurations $\alpha \subseteq_{fin} \lceil X^{\geq} \rceil$ and $\beta \subseteq_{fin} \lceil \overline{X}^{\geq} \rceil$, one has $\alpha \perp \beta$. We call a collection $\Im = \{X_i \,|\, X_i \subseteq_{fin} Init(E)\}_{i\in I}$ an *independent covering* of E if $Init(E) \subseteq \cup_{i\in I} X_i$ and each X_i is independent.

Theorem 52 Let $\mathcal{C} = (E, \leq, \rhd)$ be a DEES, let $\Im = \{X_i \,|\, X_i \subseteq_{fin} Init(E)\}_{i\in I}$ be an independent covering of E, and let $\alpha_i = \lceil X_i^{\geq} \rceil$. Then for any configuration α, $ST(\alpha) = \cup_i \alpha|X_i$. Further, $B = \{\alpha_i\}_{i\in I}$ is a basis for $\mathcal{C}$, and there are configurations $\alpha'_i \subseteq_{fin} \alpha$ such that $ST(\alpha) = \cup_i \alpha'_i$. If moreover α is finite, then $\alpha = \cup_i \alpha|X_i = \cup_i \alpha'_i$.

[4] It is interesting to note that for any redex Pu, the history of an extraction normal form of Pu [19] is internal to some finest independent set of redexes in t.

5 The Relativized Reduction Geometry

In this section, we recall some concepts and results concerning normalization relative to *stable* sets of 'partial' results, such as head-normal-forms, in DFSs [6,20]. We then relativize our Reduction Geoemtry w.r.t. stable sets of results, and prove the Optimal Decomposition Theorem: an optimal computation of a term t in a DFS, relative to a stable set of results $\mathcal{S}$, can be decomposed into, and is a sum of, optimal computations of $\mathcal{S}$-independent redex-sets of t.

Furthermore, we prove the Relative Independent Decomposition Theorem: we show that an $\mathcal{S}$-normal form of t can be computed by 'normalizing' its $\mathcal{S}$-independent redex sets, and then combining the results. Here, unlike the Optimal Decomosition Theorem, we are not restricting ourselves to complete-family reductions only. Thus our restriction throughout the paper to complete-family reductions, and to non-duplicating systems and event models, is aimed at simplifying the concepts and proofs, and is not imposed by a restricted nature of our results. The Relative Independent Decomposition Theorem appears as a natural corollary of all concepts of Relativized Reduction Geometry developed here.

Definition 53 ([4,5]) Let $\mathcal{S}$ be a set of terms in an SDRS $\mathcal{R}$.

(1) We call a redex $u \subseteq t$ $\mathcal{S}$*-needed* if at least one residual of it is contracted in any reduction from t to a term in $\mathcal{S}$, and call it $\mathcal{S}$*-unneeded* otherwise.
(2) We call a set $\mathcal{S}$ of terms *stable* if:
  (a) $\mathcal{S}$ is *closed under reduction*: $t \in \mathcal{S}$ and $t \to s$ implies $s \in \mathcal{S}$;
  (b) $\mathcal{S}$ is *closed under unneeded expansion*: for any $e \xrightarrow{u} o$ such that $e \notin \mathcal{S}$ and $o \in \mathcal{S}$, u is $\mathcal{S}$-needed.

As already mentioned in the introduction, stability of a set of 'values' is a natural sufficient condition for a normalization theory via needed reduction to be developed. The most important examples of stable sets are normal forms, head-normal forms and weak-head-normal forms in the λ-calculus, and root stable forms in orthogonal TRSs. Many more examples can be found in [20]. The following theorem combines the Relative Normalization and Optimality Theorems for DFSs [6,20]).

Theorem 54 Let $\mathcal{S}$ be a stable set of terms in a DFS $\mathcal{F}$, and let $t \notin \mathcal{S}$ be an $\mathcal{S}$-normalizable term in $\mathcal{F}$. Then t contains an $\mathcal{S}$-needed redex, and any $\mathcal{S}$-needed reduction starting from t is eventually $\mathcal{S}$-normalizing, even if finite sequences of consecutive $\mathcal{S}$-unneeded steps are also allowed. Furthermore, any $\mathcal{S}$-needed $\mathcal{S}$-normalizing complete-family reduction starting from t is shortest among $\mathcal{S}$-normalizing family-reductions starting from t.

The concepts introduced in Definition 44 (independence of reductions and redex sets, covering, basis, etc.) immediately relativize w.r.t. any stable set $\mathcal{S}$, simply by replacing 'independence', 'covering', 'basis', etc. by '$\mathcal{S}$-independence', '$\mathcal{S}$-covering', '$\mathcal{S}$-basis', etc., respectively; by replacing '(essential) redex' with '$\mathcal{S}$-needed redex', and by replacing 'maximal w.r.t. $\leq$' with $\mathcal{S}$-normalizing. In the definition of Relativized Geometry, below, we will omit definitions of $\mathcal{S}$-distance and $\mathcal{S}$-independence degree since we will not need these in the sequel. The definition then makes sense for all SDRSs.

Below $\mathcal{S}$ will always denote a stable set of terms in a DRS. Further, we will only consider $\mathcal{S}$-normalizable terms (since if a term is not $\mathcal{S}$-normalizable all redexes in it are trivially $\mathcal{S}$-needed). Note that the set of $\mathcal{S}$-normalizable terms is closed under reduction (by CR and the closure of $\mathcal{S}$ under reduction).

Definition 55 The *$\mathcal{S}$-domain* of a reduction P in a DRS, written $Dom_{\mathcal{S}}(P)$, is the set of $\mathcal{S}$-needed redexes in $MDom(P)$.

Definition 56 (Relativized Reduction Geometry) Let t be a term in an AZDFS $\mathcal{F}$.

- Two co-initial reductions $P : t \twoheadrightarrow s$ and $Q : t \twoheadrightarrow e$ are said not to $\mathcal{S}$-interact, written $P \perp_{\mathcal{S}} Q$, if $Dom_{\mathcal{S}}(P) \cap Dom_{\mathcal{S}}(Q) = \emptyset$, and any created $\mathcal{S}$-needed redex in $ft(P \sqcup Q)$ is a residual of an $\mathcal{S}$-needed redex either from $ft(P)$ or from $ft(Q)$. Otherwise, we say that P and Q (where $Dom_{\mathcal{S}}(P) \cap Dom_{\mathcal{S}}(Q) = \emptyset$) *$\mathcal{S}$-interact*, $P \not\perp_{\mathcal{S}} Q$.
- We call a set $\Pi = \{P_i\}_{i \in I}$ of reductions starting from t *$\mathcal{S}$-independent* if $P'_i \perp_{\mathcal{S}} \sqcup_{i \neq j} P'_j$ for every $i \in I$ and any $P'_i \leq_{fin} P_i$. We call Π an *$\mathcal{S}$-basis* of $\mathcal{F}$ at t if Π is $\mathcal{S}$-independent and $\sqcup_{i \in I} P_i$ is $\mathcal{S}$-normalizing.
- We call $U \subseteq t$ *$\mathcal{S}$-independent* if no (finite) U-reduction can $\mathcal{S}$-interact with an $\overline{U}$-reduction. We call $\Im = \{U_i\}_{i \in I}$ an *$\mathcal{S}$-independent covering* of t if $\Im$ is a covering of the set $U_{\mathcal{S}}(t)$ of $\mathcal{S}$-needed redexes in t and each U_i is $\mathcal{S}$-independent.

It is easy to see that $Dom_{\mathcal{S}}(P) = SDom(P) \cap U_{\mathcal{S}}(t) = Dom(P) \cap U_{\mathcal{S}}(t)$. This follows from the following lemma:

Lemma 57 Let $u \subseteq t \stackrel{P}{\twoheadrightarrow}$, in an AZDFS, and let u be P-erased and P-unneeded. Then it is $\mathcal{S}$-unneeded (i.e., every step of P is $\mathcal{S}$-unneeded).

Proof Let $P^* \approx_L P$ be external to u, and let $P' : t \twoheadrightarrow s$ be a finite initial part of P^* that erases u. If P' is $\mathcal{S}$-normalizing, then u is clearly $\mathcal{S}$-unneeded (by Definition 53); otherwise, for any $\mathcal{S}$-normalizing reduction $P'' : s \twoheadrightarrow e$, $P' + P''$ is $\mathcal{S}$-normalizing and external to u, thus again u is $\mathcal{S}$-unneeded (recall that any reduct of an $\mathcal{S}$-normalizable term remains $\mathcal{S}$-normalizable).

In [6,20], we defined an algorithm which associates with a reduction P an $\mathcal{S}$-needed reduction $[P]_{\mathcal{S}}$ such that $P \approx_L [P]_{\mathcal{S}} + [P]^{-}_{\mathcal{S}}$, where $[P]^{-}_{\mathcal{S}}$ is $\mathcal{S}$-unneeded. The algorithm works for *regular* stable sets of terms $\mathcal{S}$ in any SDRS, where regular means that $\mathcal{S}$-unneeded redexes cannot duplicate $\mathcal{S}$-needed ones. Thus it applies for any stable set $\mathcal{S}$ in an AZDFS, and for any P one has $FAM_{\mathcal{S}}(P) = FAM([P]_{\mathcal{S}})$. The algorithm simply pushes all $\mathcal{S}$-needed steps of P before $\mathcal{S}$-unneeded steps, which is possible by the following lemma:

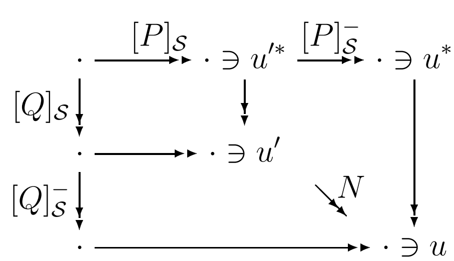
Lemma 58 ([6,20]) Let $t \notin \mathcal{S}$, in a DFS $\mathcal{F}$.

(1) Residuals of $\mathcal{S}$-unneeded redexes in t remain $\mathcal{S}$-unneeded.
(2) Let $t \xrightarrow{u} t'$, let u be $\mathcal{S}$-unneeded, and let $u' \subseteq t'$ be a created redex. Then u' is $\mathcal{S}$-unneeded.
(3) Let $v \subseteq t$ be $\mathcal{S}$-needed, let $t \xrightarrow{u} s$, and let $v \neq u$. Then v has an $\mathcal{S}$-needed residual in s.

Since any $\mathcal{S}$-needed reduction P in an AZDFS is standard by Lemma 57, $SDom([P]_{\mathcal{S}}) = MDom([P]_{\mathcal{S}})$. Using this observation, we can give an equivalent definition of $\mathcal{S}$-interaction using $\mathcal{S}$-needed reductions:

Lemma 59 Let P, Q be finite co-initial reductions in an AZDFS. Then P and Q $\mathcal{S}$-interact iff $[P]_{\mathcal{S}}$ and $[Q]_{\mathcal{S}}$ do.

Proof We need to show that $P \perp_{\mathcal{S}} Q$ iff $MDom([P]_{\mathcal{S}}) \cap MDom([Q]_{\mathcal{S}}) = \emptyset$ and any created $\mathcal{S}$-needed redex in $ft([P]_{\mathcal{S}} \sqcup [Q]_{\mathcal{S}})$ is a residual of an $\mathcal{S}$-needed redex created by $[P]_{\mathcal{S}}$ or by $[Q]_{\mathcal{S}}$. Indeed, $Dom_{\mathcal{S}}(P) = MDom(P) \cap U_{\mathcal{S}}(t) = SDom(P) \cap U_{\mathcal{S}}(t) =$ (by definition of $[P]_{\mathcal{S}}$) $= SDom([P]_{\mathcal{S}}) =$ (since $[P]_{\mathcal{S}}$ is standard) $= MDom([P]_{\mathcal{S}})$, hence $Dom_{\mathcal{S}}(P) \cap Dom_{\mathcal{S}}(Q) = \emptyset$ iff $MDom([P]_{\mathcal{S}}) \cap MDom([Q]_{\mathcal{S}}) = \emptyset$. Further, if $ft(P \sqcup Q)$ contains a created $\mathcal{S}$-needed redex u, then by Lemma 58.(1)-(2) u must be a residual of an $\mathcal{S}$-needed redex $u' \in ft([P]_{\mathcal{S}} \sqcup [Q]_{\mathcal{S}})$ (since $P \sqcup Q \approx_L [P]_{\mathcal{S}} \sqcup [Q]_{\mathcal{S}} + N$ for some $\mathcal{S}$-unneeded N, see the figure below). Conversely, any $\mathcal{S}$-needed redex $u' \in ft([P]_{\mathcal{S}} \sqcup [Q]_{\mathcal{S}})$ has an $\mathcal{S}$-needed N-residual $u \in ft(P \sqcup Q)$ by Lemma 58.(3). Finally, if u is a residual of a redex u^* say in $ft(P)$, then by Lemma 58.(1)-(2) u^* is $\mathcal{S}$-needed and is a residual of an $\mathcal{S}$-needed $u'^* \in ft([P]_{\mathcal{S}})$, and u' is a residual of u'^* by Proposition 7 (as u' and u'^* are zig-zag related). Conversely, if an $\mathcal{S}$-needed redex $u' \in ft([P]_{\mathcal{S}} \sqcup [Q]_{\mathcal{S}})$ is a residual of $u'^* \in ft([P]_{\mathcal{S}})$, then by Lemma 58 u'^* is $\mathcal{S}$-needed, it has an $\mathcal{S}$-needed residual $u^* \in ft(P)$, u' has an $\mathcal{S}$-needed N-residual $u \in ft(P \sqcup Q)$, and u is a residual of u^* by Proposition 7, and we are done.



Similar reformulations can be given for the other concepts of the relativized geometry.

Note that the relativized independence concept allows in general for a finer independent covering of a term, and hence allows for more independence in computation. For example, let $\mathcal{S}$ be the set of head-normal forms, in the λ-calculus, and let $t = (\lambda x.(\lambda y.x''y)((\lambda x'.x)ab))((\lambda z.(\lambda z'.z''))c)$. t contains four redexes, $u_1 = t, u_2, u_3$ and u_4, underlined and enumerated from left to right. u_1 and u_2 are on the left-spine, and are $\mathcal{S}$-needed, i.e., head-needed. Their contraction yields a head-normal form (no new head-needed redexes are created), therefore $U_h = \{\{u_1\}, \{u_2\}\}$ is a head-independent covering of t. However, contraction of all four redexes creates a new redex $(\lambda z'.z'')b$ in $x''((\lambda z'.z'')b)$, and therefore no proper subset of $U = \{u_1, u_2, u_3, u_4\}$ is independent, i.e., $\{U\}$ is the only independent covering of t.

Lemma 60 In an AZDFS , $U \subseteq t$ is $\mathcal{S}$-independent iff any $\mathcal{S}$-needed reduction $P : t \twoheadrightarrow$ respects it.

Proof First note that P is finite by Theorem 54 (since we only consider $\mathcal{S}$-normalizable terms t), and it is standard by Lemma 57.

($\Rightarrow$) Suppose on the contrary that not every $\mathcal{S}$-needed reduction starting from t respects U, and let $P = P' + u$ be a shortest one. Since P' respects U, we have that $P' \approx_{STA} P'|U \sqcup P'|\overline{U}$ by Lemma 28. Since P does not respect U, u is not the residual of a redex either in $ft(P'|U)$ or in $ft(P'|\overline{U})$. But this means that $P'|U$ and $P'|\overline{U}$ $\mathcal{S}$-interact (since by the Cube Lemma u is a created redex, and it is $\mathcal{S}$-needed), i.e., U is not $\mathcal{S}$-independent.

($\Leftarrow$) Let U not be $\mathcal{S}$-independent. Then there are reductions N and Q respectively internal and external to U that $\mathcal{S}$-interact, i.e., there is a created $\mathcal{S}$-needed redex u in $ft(N \sqcup Q)$ that is not a residual of a redex either in $ft(N)$ or in $ft(Q)$. By Lemma 59, N and Q can be chosen $\mathcal{S}$-needed, and so is $N \sqcup Q$ by Lemma 58.(3). If $N \sqcup Q$ does not respect U, then we are done. Otherwise, say $(N \sqcup Q) + u$ does not respect U.

We will need the following relativized version of U-fairness:

Definition 61 Let $U \subseteq t$, in a DRS $\mathcal{R}$. We call a U-reduction $P : t \twoheadrightarrow s$ $(U, \mathcal{S})$-fair if s does not contain $\mathcal{S}$-needed U-redexes. (Note that s may contain $\mathcal{S}$-unneeded U-redexes, thus P need not be U-fair.)

Lemma 62 Let $\Im = \{U_i \,|\, i \in I\}$ be an $\mathcal{S}$-independent covering of t, in an AZDFS.

(1) If $P : t \twoheadrightarrow s$ is an $\mathcal{S}$-normalizing $\mathcal{S}$-needed reduction, then $P_i = P|U_i$: $t \twoheadrightarrow s_i$ are $(U_i, \mathcal{S})$-fair $\mathcal{S}$-needed U_i-reductions.

(2) If $P_i : t \twoheadrightarrow s_i$ are $(U_i, \mathcal{S})$-fair $\mathcal{S}$-needed U_i-reductions, then $P = \sqcup_{i \in I} P_i$:

$t \twoheadrightarrow s$ is an $\mathcal{S}$-normalizing $\mathcal{S}$-needed reduction.

Proof

(1) By Definition 27, P_i is an U_i-reduction, and it contracts residuals of redexes contracted in P, hence is $\mathcal{S}$-needed by Lemma 58.(3). Suppose on the contrary that say P_j is not $(U_j, \mathcal{S})$-fair, i.e., there is $v \subseteq s_j$ that is an U_j-redex and is $\mathcal{S}$-needed. Let $\overline{P_j} = P|\overline{U_j}$. By Lemma 60, P respects U_j (and $\overline{U_j}$), hence by the Decomposition Theorem $P \approx_L P_j \sqcup \overline{P_j}$. Hence $\overline{P_j}/P_j : t \twoheadrightarrow s$ and it is external to v, thus by Lemma 58.(3) v has an $\mathcal{S}$-needed residual in s – contradiction, since $s \in \mathcal{S}$.

(2) Since every P_i is $\mathcal{S}$-needed, so is P by Lemma 58.(3). Since $\Im = \{U_i \mid i \in I\}$ is an $\mathcal{S}$-independent covering of t and $P_i \sqcup \overline{P_i} \approx_L P$, where $\overline{P_i} = \sqcup_{j \neq i} P_j$, every $\mathcal{S}$-needed redex in s is a $\overline{P_i}/P_i$-residual of an $\mathcal{S}$-needed (by Lemma 58.(1)) redex in s_i, for some i (it can be shown using Proposition 30 that $\overline{P_i} \approx_L P|\overline{U_i}$). Since every P_i is an $(U_i, \mathcal{S})$-fair U_i-reduction, any $\mathcal{S}$-needed redex $u \subseteq s_i$ is a residual of an $\mathcal{S}$-needed (by Lemma 58.(1)) redex $v \in U_j$ for some $j \neq i$. But since P_j is $(U_j, \mathcal{S})$-fair, v does not have P_j-residuals. Hence, by the Cube Lemma, u does not have $\overline{P_i}/P_i$-residuals. Thus s does not contain $\mathcal{S}$-unneeded redexes, and by Theorem 54, $s \in \mathcal{S}$.

Lemma 63 Let $P : t \twoheadrightarrow s$ be an $(U, \mathcal{S})$-fair U-reduction, let $t \xrightarrow{u} e$, where $u \in U \subseteq t$ and let U' be the set of U-redexes in e. Then $P' = P/u : e \twoheadrightarrow o$ is an $(U', \mathcal{S})$-fair U'-reduction.

Proof Since P is $(U, \mathcal{S})$-fair, there are no $\mathcal{S}$-needed U-redexes in s. Thus if u has a P-residual in s, it is $\mathcal{S}$-unneeded, therefore, by Lemma 58.(2), all the redexes in o created by u/P are $\mathcal{S}$-unneeded. Again by Lemma 58.(1), u/P-residuals of U-redexes in s remain $\mathcal{S}$-unneeded. Thus any U-redex in o is $\mathcal{S}$-unneeded. But the sets of U-redexes and U'-redexes in o coincide by the Cube Lemma, because both sets consist precisely of redexes *not* in $\overline{U}/(u \sqcup P)$. Hence P' is an $(U', \mathcal{S})$-fair U'-reduction.

Lemma 64 Let $U \subseteq t$ contain an $\mathcal{S}$-needed redex, in an AZDFS $\mathcal{F}$. A reduction starting from t is a shortest $(U, \mathcal{S})$-fair U-reduction iff it is an $\mathcal{S}$-needed $(U, \mathcal{S})$-fair U-reduction.

Proof

($\Leftarrow$) Let Q and P be $(U, \mathcal{S})$-fair U-reductions, and let Q be $\mathcal{S}$-needed. We show $|Q| \leq |P|$ by induction on $|Q|$. Since U contains an $\mathcal{S}$-needed redex, $|Q|, |P| \neq 0$. So let $Q : t \xrightarrow{v} s \overset{Q'}{\twoheadrightarrow} e$. Since v is $\mathcal{S}$-needed and P is $(U, \mathcal{S})$-fair, it follows from Lemma 58.(3) that the residuals of v along P are $\mathcal{S}$-needed, hence P must contract one of them. Hence, by affineness, $|P'| \leq |P| - 1$, where $P' = P/v$. Let U' be the set of U-redexes in s. Then, by Lemma 63, P'

and Q' are $(U', \mathcal{S})$-fair U'-reductions. Hence, by the induction assumption, $|Q'| \leq |P'|$. Thus, $|Q| \leq |P|$.

($\Rightarrow$) Let Q and P be $(U, \mathcal{S})$-fair U-reductions, let Q be $\mathcal{S}$-needed, but P not be $\mathcal{S}$-needed. We show that $|Q| < |P|$ by induction on $|P|$. So let $P = u + P'$ (we already know that $|Q|, |P| \neq 0$). Since Q is $\mathcal{S}$-needed, so is $Q' = Q/u$ by Lemma 58.(3). Suppose first that u is $\mathcal{S}$-unneeded. Then so is every residual of it along Q by Lemma 58.(1). Hence Q is external to u. Since $\mathcal{S}$-unneeded redexes cannot erase $\mathcal{S}$-needed ones, $|Q'| = |Q|$. But, by ($\Leftarrow$), $|Q'| \leq |P'|$, since both Q' and P' are $(U', \mathcal{S})$-fair U'-reductions by Lemma 63, where U' is the set of U-redexes in $ft(u)$, and Q' is $\mathcal{S}$-needed. Hence $|Q| < |P|$ in that case. Now assume u is $\mathcal{S}$-needed. Then $|Q'| = |Q| - 1$, as we have shown above (in the proof of ($\Leftarrow$)). Thus, by the induction assumption, $|Q'| < |P'|$, and we have again $|Q| < |P|$.

Now, as an immediate consequence of Lemma 62, Theorem 54, and Lemma 64, we have the following Optimal Decomposition Theorem for AZDFSs:

Theorem 65 Let $\mathcal{S}$ be a stable set of terms in an AZDFS $\mathcal{F}$, let $\Im = \{U_i\}_{i \in I}$ be an $\mathcal{S}$-independent covering of an $\mathcal{S}$-normalizable term t in $\mathcal{F}$, let $\Im' = \{U_j\}_{j \in J \subseteq I}$ contain all U_i that contain at least one $\mathcal{S}$-needed redex of t, and let P_j be internal to U_j. Then P_j are optimal (i.e., shortest) $(U_j, \mathcal{S})$-fair reductions iff $P = \sqcup_j P_j$ is an optimal $\mathcal{S}$-normalizing reduction starting from t.

It is shown in [19,20] that $\mathcal{S}$-needed complete family-reductions in a DFS $\mathcal{F}$ precisely correspond to $\mathcal{S}_I$-needed reductions in the corresponding AZDFS $\mathcal{F}_I$, where $\mathcal{S}_I = \mathcal{S} \cap Ter(\mathcal{F}_I)$ is stable in $\mathcal{F}_I$ ($Ter(\mathcal{F}_I)$ is the set of terms in $\mathcal{F}_I$). Hence we have immediately from Remark 6 and Theorem 65 the following Optimal Decomposition Theorem for DFSs:

Corollary 66 (Optimal Decomposition) Let $\mathcal{S}$ be a stable set of terms in a DFS $\mathcal{F}$, let $\Im = \{U_i\}_{i \in I}$ be an $\mathcal{S}$-independent covering of an $\mathcal{S}$-normalizable term t in $\mathcal{F}$, let $\Im' = \{U_j\}_{j \in J \subseteq I}$ contain all U_i that contain at least one $\mathcal{S}$-needed redex of t, and let P_j be complete family-reductions internal to U_j. Then P_j are optimal $(U_j, \mathcal{S})$-fair complete family-reductions iff $P = \sqcup_j P_j$ is an optimal $\mathcal{S}$-normalizing complete family-reduction starting from t.

Given a reduction $N : t_0 \xrightarrow{v_0} t_1 \xrightarrow{v_1} \ldots t_n$ in a DFS $\mathcal{F}$, one can construct its *corresponding* complete family-reduction $N^c : t_0 = s_0 \xrightarrow{V_0}\!\!\!\!\to s_1 \xrightarrow{V_1}\!\!\!\!\to \ldots s_n$, where $Fam(v_k) = Fam(V_k)$ for all $k = 1, \ldots, n$; $\exists N'_k : t_k \twoheadrightarrow s_k$ such that $N^c_k \approx_L N_k + N'_n$, and N^c_k and N_k are the initial parts of N^c and N of length k; and V_k contains all residuals of v_k in s_k, and possibly other redexes too. (Because of erasure of redexes, some of V_k may be empty if v_k is not $\mathcal{S}$-needed.) Hence $FAM(N) \subseteq FAM(N^c)$. Furthermore, if N is internal to $W \subseteq t_0$, then so is N^c, and if N is $(W, \mathcal{S})$-fair, so is N^c, and the reduction $N'_n : t_n \twoheadrightarrow s_n$, henceforth denoted N^-, is $\mathcal{S}$-unneeded.

Using the above observations, we now show that one does not need to restrict to complete family-reductions in order to be able to compute independent redex sets of an $\mathcal{S}$-normalizing term in parallel:

Theorem 67 (Relative Independent Decomposition) Let $\mathcal{S}$ be a stable set of terms in a DFS $\mathcal{F}$, let $\Im = \{U_i\}_{i \in I}$ be an $\mathcal{S}$-independent covering of an $\mathcal{S}$-normalizable term t in $\mathcal{F}$, let $\Im' = \{U_j\}_{j \in J \subseteq I}$ contain all U_i that contain at least one $\mathcal{S}$-needed redex of t, and let Q_j be reductions internal to U_j. Then Q_j are $(U_j, \mathcal{S})$-fair reductions iff $\sqcup_j Q_j$ is an $\mathcal{S}$-normalizing reduction starting from t.

Proof

- **(⇒)** Since all Q_j are (respectively) $(U_j, \mathcal{S})$-fair, so are all Q_j^c, and hence $\sqcup_{j \in J} Q_j^c$ is $\mathcal{S}$-normalizing by Corollary 66. But $Q_j^c \approx_L Q_j + Q_j^-$, where Q_j^- are $\mathcal{S}$-unneeded, and by Lemma 58 $\sqcup_{j \in J} Q_j^c \approx_L \sqcup_{j \in J} Q_j + Q'$ for some $\mathcal{S}$-unneeded Q'. Since $\mathcal{S}$-unneeded steps cannot enter $\mathcal{S}$, $\sqcup_{j \in J} Q_j$ is $\mathcal{S}$-normalizing.
- **(⇐)** If at least one Q_{j_0} is not $(U_{j_0}, \mathcal{S})$-fair, then $ft(Q_{j_0})$ contains an $\mathcal{S}$-needed U_{j_0}-redex, which must have an $\mathcal{S}$-needed residual in $ft(\sqcup_j Q_j)$ by Lemma 58 (since by the family axioms Q_j contract redexes in different families). Hence $ft(\sqcup_j Q_j) \notin \mathcal{S}$ by Theorem 54.

As we have already remarked above, the definition of $\mathcal{S}$-independence in Definition 56 makes sense for any co-initial reductions in DFSs (not only for complete family-reductions). It is easy to see that $N \perp_{\mathcal{S}} P$ iff $N^c \perp_{\mathcal{S}} P^c$ (since every jointly created $\mathcal{S}$-needed redex in $ft(N \sqcup P)$, if any, has an $\mathcal{S}$-needed residual in $ft(N^c \sqcup P^c)$, by the construction of N^c and P^c). Thus, in the above theorem, $\Im = \{U_i\}_{i \in I}$ is an $\mathcal{S}$-independent covering of t iff no reduction internal to U_i interacts with a reduction external to U_i, for every $i \in I$. Using this fact, a more direct proof of Theorem 67, similar to the proof of Lemma 62, can also be given.

6 Conclusions

We have defined concepts similar to those in *Vector Spaces* for orthogonal rewrite systems, and described how these can be used in distributed evaluation of sequential programs. The constructed *Reduction Geometry* is not just a nice piece of mathematics. Obviously, (relative) independence of redex-sets is undecidable in general, as is neededness. However, we hope that decidable approximations for independence can be defined which will yield decidable concepts for large classes of rewrite systems, as is the case for the neededness [8].

For example, all the introduced concepts are decidable for *Recursive Program Schemes* (RPSs), both in first [14] and higher order [15] cases. Several kinds of RPS were extensively studied in the literature, mainly in the seventies [3]. Our (first-order) RPSs correspond to *Applicative RPSs* in [3]. The left-hand sides of RPS-rules contain one defined symbol only, and RPSs do not have full computational power as the $if - then - else$ operator is only evaluated semantically (i.e., there is no rewrite rule for it). Actually, because of a specific simple form of redex-creation in such systems – created redexes are determined by the rewrite rules only, and the arguments or the context in which a redex is contracted do not contribute to redex-creation – one has maximal possible independence there: any redex forms an independent redex-set. It is shown (very briefly) in [15] how computation of the normal form of a term t in a higher-order RPS can be decomposed into (concurrent) computation of its redexes, and how the latter can be combined to yield the normal form of t. (For first-order RPSs this is much simpler since there are no β-like substitution steps involved.) This technique even yields a method of transforming RPSs into simpler *irreducible* ones where the right-hand sides of rules are in normal form. All these are based on the concept of *essential similarity* of redexes, which contains a 'minimal information' determining the normalization behaviour of redexes in higher-order RPSs (essentiality is a refinement of neededness that makes sense for all subterms, not for redexes only).

Another important example where independent redex-sets can be found effectively is the set of *hyperbalanced* λ-terms [23,24]. Hyperbalanced terms form the smallest subset of λ-terms, closed under β-reduction, such that for any maximal subterm of the form $(\lambda x_1 \ldots x_n.s)t_1 \ldots t_m$ one has $m = n$, and whenever t_1 is an abstraction $\lambda y_1 \ldots y_l.o$, every free occurrence of x_1 in s is in the subterm of the form $x_1 o_1 \ldots o_l$. All simply typable terms, up to a restricted η-expansion, are hyperbalanced. All hyperbalanced terms are strongly normalizable, and in such terms all unneeded (or rather, inessential) subterms can be statically detected. Among other things, these properties arise from the fact that only two, of the three kinds of redex-creation in the λ-calculus [28], are possible in the setting of hyperbalanced terms. As in RPSs, any redex in a hyperbalanced term forms an independent redex-set [12], and our theory suggests that they can be evaluated independently – every redex uses only a substructure of a given term, and the substructures for different redexes do not 'overlap'. However, unlike RPSs, these substructures are not simply occurrences of contexts, and an appropriate implementation technique for evaluating redexes independently from each other in hyperbalanced terms is not known at present.

Finally, we remark that there is an obvious relationship between the independence concept and modularity of properties in TRSs (see [26] for a survey) which we think is worth investigating. For example, in the simple case where two orthogonal rewrite systems R_1 and R_2 have disjoint alphabets Σ_1 and Σ_2, and their rules are non-collapsing (i.e., right-hand sides contain a function symbol), then in any term constructed from symbols in $\Sigma_1 \cup \Sigma_2$, the sets consisting of respectively R_1-redexes and R_2-redexes are independent, therefore for example SN is a modular property: $R_1 \cup R_2$ is SN iff R_1 and R_2 are SN. This result for (not-necessarily orthogonal) TRSs is proved in [35].

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