An Abstract Böhm-normalization

John Glauert [1]

School of Information Systems, UEA
Norwich NR4 7TJ, UK

Zurab Khasidashvili [2]

Department of CS, Bar-Ilan University
Ramat-Gan 52900, Israel

Abstract

We study *normalization by neededness* with respect to 'infinite results', such as Böhm-trees, in an abstract framework of *Stable Deterministic Residual Structures.* We formalize 'infinite results' as suitable sets of infinite reductions, and prove an abstract infinitary normalization theorem with respect to such sets. We also give a sufficient and necessary condition for existence of *minimal* normalizing reductions.

1 Introduction

A normalizable term in a rewriting system may have an infinite reduction, so it is important to have a *normalizing* strategy which enables one to construct reductions to normal form. For example, it is well known that the leftmost-outermost strategy is normalizing in the λ-calculus [12]. For Orthogonal TRSs, Huet and Lévy [19] found a general normalizing strategy that always contracts a *needed* redex, one with a residual that has to be contracted in any reduction to normal form.

This fundamental work on neededness has been extended in several directions. Barendregt et al. [6], Maranget [34], Nöcker [38] and Middeldorp [37] studied neededness w.r.t. head-normal forms, weak head-normal forms, constructor head-normal forms, and root-stable forms, respectively. Kennaway et al. [22] studied a needed strategy for infinitary orthogonal TRSs. A different approach to normalization is developed in Kennaway [20] and Antoy and Middeldorp [3]. Khasidashvili [23,24] introduced a refinement of the needed

[1] Email: `jrwg@sys.uea.ac.uk`
[2] Email: `khasidz@cs.biu.ac.il`

strategy which is based on a concept of descendant which refines that of residual. Melliès [36] and van Oostrom [40] extended the needed and essential strategies, respectively, to some non-orthogonal systems.

In [15], the present authors addressed the question of normalization relative to a desired set of final terms, considering the properties that a set $\mathcal{S}$ of terms must possess in order for the neededness theory of Huet and Lévy still to make sense. This work is done in the context of orthogonal *Expression Reduction Systems* (ERSs), a formalism for rewriting which subsumes Term Rewriting and the λ-calculus. Natural conditions were imposed on $\mathcal{S}$, called *stability*, that are sufficient for the following *Relative Normalization* (RN) theorem to hold: each $\mathcal{S}$-normalizable term not in $\mathcal{S}$ (that is, not in $\mathcal{S}$*-normal form*) has at least one $\mathcal{S}$-needed redex, and repeated contraction of such redexes always leads to an $\mathcal{S}$-normal form. It is shown also that if a stable $\mathcal{S}$ is *regular*, i.e., if $\mathcal{S}$-unneeded redexes cannot duplicate $\mathcal{S}$-needed ones, then the $\mathcal{S}$-needed strategy is hypernormalizing as well, and *minimal* (w.r.t. the Lévy-embedding relation) $\mathcal{S}$-normalizing reductions exist.

In [14], we further generalized the theory by abstracting from the concrete structure of terms. We studied relative normalization in Deterministic Residual Structures (DRSs), which are abstract reduction systems with an axiomatized residual relation. Despite their highly abstract nature, a counterpart of the *stability* property of Berry [8] and Winskel [44] enabled us to prove the RN theorem for all *regular* stable sets $\mathcal{S}$. (We actually prove the Relative Hypernormalization theorem.) We show that without this stability axiom the theorem fails. The proof method employed is similar to that in [23,24], and is based on the fact that $\mathcal{S}$-needed steps in a reduction can be pushed before $\mathcal{S}$-unneeded steps without affecting the number of $\mathcal{S}$-needed steps. In [26], we studied *discrete normalization* in SDRSs, which is normalization relative to particular (finite or infinite) reductions.

Here we extend the concepts of stability and regularity from sets of terms (which represent 'finite' results, or values, such as head normal forms) to sets of reductions, to represent the concepts of 'infinite' values, such as Böhm trees. Throughout the paper, we restrict ourselves to finite terms – terms can contain only a finite number of redexes. For regular stable sets $\mathcal{S}$ of finite or infinite results and an $\mathcal{S}$-normalizable term t, in an SDRS, we show that all $\mathcal{S}$-needed fair reductions starting from t are $\mathcal{S}$-normalizing, and if t has a finite $\mathcal{S}$-normalizing reduction, then any $\mathcal{S}$-needed reduction starting from t eventually normalizes t in a finite number of steps. This result unifies the Relative Hypernormalization theorem with the Discrete Normalization theory. We also show that for $\mathcal{S}$-minimal reductions to exist, for regular stable sets of finite results $\mathcal{S}$, every $\mathcal{S}$-normalizing term must possess an $\mathcal{S}$-needed $\mathcal{S}$-erased redex (where a redex is $\mathcal{S}$-erased if it does not have a residual under $\mathcal{S}$-normalizing reductions).

The paper is organized as follows. In the next section, we recall SDRSs and present some fundamental lemmas concerning (mutually) *external* reductions. In section 3, we give a construction of the meet operation with respect to

Levý's ordering on reductions. In Section 4, we introduce the concept of stability for sets of reductions, and give examples. In Section 5, we prove the RN theorem for regular stable sets of reductions in SDRSs, and present the minimality results. Conclusions appear in Section 6.

2 Deterministic Residual Structures

In this section we recall *Deterministic Residual Structures* (DRSs) which are *Abstract Reduction Systems* (ARSs) satisfying certain properties concerning *residuals*. Residuals of redexes were first introduced and studied by Church and Rosser in the λI-calculus [11]. The study of abstract systems with residuals started from Hindley [17].

The definition and some results about ARSs can be found e.g., in [29,18]. Our definition is slightly different.

Definition 2.1 An ARS is a triple $A = (Ter, Red, \rightarrow)$ where Ter is a set of *terms*, ranged over by t, s, o, e; Red is a set of *redexes* (or *redex occurrences*), ranged over by u, v, w; and $\rightarrow$: $Red \rightarrow (Ter \times Ter)$ is a function such that for any $t \in Ter$ there is only a finite set of $u \in Red$ such that $\rightarrow(u) = (t, s)$, written $t \stackrel{u}{\rightarrow} s$. This set will be known as the redexes of term t, where $u \in t$ denotes that u is a member of the redexes of t and $U \subseteq t$ denotes that U is a subset of the redexes. Note that $\rightarrow$ is a *total* function, so one can identify u with the triple $t \stackrel{u}{\rightarrow} s$.

A *reduction* is a sequence $t \stackrel{u_1}{\rightarrow} t_2 \stackrel{u_2}{\rightarrow} \ldots$. Reductions are denoted by P, Q, N. We write $P : t \twoheadrightarrow s$ or $t \stackrel{P}{\twoheadrightarrow} s$ if P denotes a reduction (sequence) from t to s. $Q : t \twoheadrightarrow$ may be finite as well as infinite. Reductions are *co-initial* if they share the same initial term. $P + Q$ denotes the concatenation of P and Q. u also denotes the reduction that contracts u. Further, for any reduction Q, $\lceil Q \rceil^k$ will denote the initial part of Q of length k, provided $k \leq |Q|$ (the length of Q), and $\lceil Q \rceil_k$ will denote the tail of Q starting from the $(k+1)$th step; thus $Q = \lceil Q \rceil^k + \lceil Q \rceil_k$. Finally, we write $P \leq Q$ if P is an initial part of Q.

Definition 2.2 (Deterministic Residual Structure) A *Deterministic Residual Structure* (DRS) is a pair $\mathcal{R} = (A, /)$, where A is an ARS and $/$ is a *residual* relation on redexes relating redexes in the source and target term of every reduction $t \stackrel{u}{\rightarrow} s \in A$, such that for $v \in t$, the set v/u of *residuals of v under u* is a set of redexes of s; a redex in s may be a residual of only one redex in t under u, and $u/u = \emptyset$. If v has more than one u-residual, then *u duplicates v*. If $v/u = \emptyset$, then *u erases v*, and if moreover $v \neq u$, then u *discards* v. A redex of s which is not a residual of any $v \in t$ under u is said to be *u-new* or *created* by u. The set v/P of residuals of v under any finite reduction P is defined by transitivity. $v/P = \{v\}$ if $|P| = 0$.

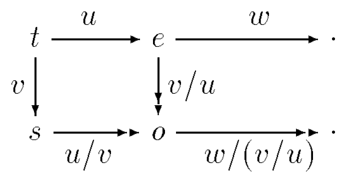
A *development* of $U \subseteq t$ is a reduction $P : t \twoheadrightarrow$ that only contracts residuals of redexes from U; it is *complete* if it is finite and $U/P = \cup_{u \in U} u/P = \emptyset$. Development of $\emptyset$ is identified with the empty reduction. The residual relation satisfies the following two axioms:

[FD] (*Finite Developments* [16]) All developments are terminating; all complete developments of $U \subseteq t$ end at the same term; and residuals of a redex $v \in t$ under all complete developments of U are the same. Below U will also denote a complete development of $U \subseteq t$.

[weak acyclicity] ([43]) Let $u, v \in t$, $u \neq v$, and $u/v = \emptyset$. Then $v/u \neq \emptyset$.[3]

We call a DRS $\mathcal{R}$ *stable* (SDRS) if the following axiom is satisfied:

[stability] If $u, v \in t$ are different redexes, $t \overset{u}{\to} e$, $t \overset{v}{\to} s$, and u creates a redex $w \in e$, then the redexes in $w/(v/u)$ are not u/v-residuals of redexes of s, i.e., they are created along u/v.



DRSs are closely related to *Determinate Concurrent Transition Systems* (DCTS) of Stark [43], and to *Abstract Reduction Systems* (ARSs) of Gonthier et al. [16]. The main difference from DCTSs is that Stark requires a non-duplicating residual relation. Unlike ARSs of [16], we do not have a nesting relation on redexes and the corresponding axioms, and the stability axiom is modified accordingly. Instead, we study properties of conflict-free transition and reduction systems based on the duplication and erasure effect of executed transitions on the others, and develop a theory that is applicable to systems with nested transitions too. Other related abstract reduction systems are studied in [35,39,41].

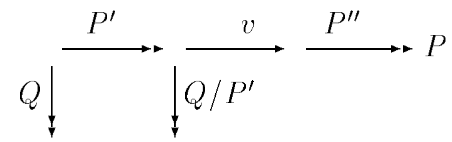
One can verify that all orthogonal (first or fully extended higher-order, see e.g., [41]) term rewriting systems, such as the λ-calculus, form DRSs. These systems are stable, and can be shown so just using an appropriate notion of descendant which assigns the contractum to the contracted redex – labelling is not necessary. Further, orthogonal Term Graph Rewriting Systems [21] are DRSs but they do not satisfy all the nesting axioms of [16].

The properties of the residual relation in orthogonal systems are all standard [19,31,9,32,10,43,35], and we only review quickly the main constructions used in this paper. In a DRS $\mathcal{R}$, *Lévy-equivalence* or *permutation-equivalence* is defined as the smallest relation on co-initial finite reductions satisfying: $U+V/U \approx_L V+U/V$ and $Q \approx_L Q' \;\Rightarrow\; P+Q+N \approx_L P+Q'+N$, where U and V are sets of redexes in the same term. The residual relation on redexes is extended to all co-initial reductions as follows: $(P_1+P_2)/Q = P_1/Q+P_2/(Q/P_1)$ and $Q/(P_1+P_2) = (Q/P_1)/P_2$. The *Lévy-embedding* relation on co-initial finite reductions, $\trianglelefteq_L$, is defined by: $P \trianglelefteq_L Q$ iff $P/Q = \emptyset$. One can show that $P \approx_L Q$ iff $P \trianglelefteq_L Q$ and $Q \trianglelefteq_L P$ and that $P \trianglelefteq_L Q$ iff $Q \approx_L P + N$ for some

[3] This axiom is called [acyclicity] in [14], and is axiom (4) in [43].

N. Intuitively, $P \trianglelefteq_L Q$ expresses that Q does more work than P, and Q/P is the part of Q that remains after P has been done. Further, one can prove the following *strong Church-Rosser* theorem for DRSs: For all co-initial finite reductions P and Q in a DRS, $P \sqcup Q \approx_L Q \sqcup P$, where $P \sqcup Q$ abbreviates $P + Q/P$. And finally, we will also need the following *Cube Lemma*: for any finite co-initial reductions P, Q and N, $N/(P \sqcup Q) = N/(Q \sqcup P)$.

Lévy-equivalence extends to infinite reductions as follows. First, for any $u \in t \stackrel{P}{\twoheadrightarrow}$, define $u/P = \emptyset$ if $u/P' = \emptyset$ for some finite $P' \leq P$ (we say u is *erased* in P). Now, for any co-initial P, Q, define $P/Q = \emptyset$, or equivalently, $P \trianglelefteq_L Q$, if for any redex v contracted in P, say $P = P' + v + P''$, $v/(Q/P') = \emptyset$ (see the diagram); and define $P \approx_L Q$ iff $P \trianglelefteq_L Q$ and $Q \trianglelefteq_L P$.

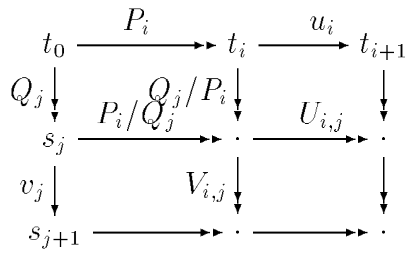


Finally, P/Q is only defined for finite Q, as the reduction whose initial parts are residuals of the initial parts of P under Q.

The [stability] axiom is not used in the above constructions. Intuitively, stability means that a redex cannot arise from two unrelated sources. This property has a natural extension to many-step reductions, where 'unrelated' is formalized by the notion of *external* which captures the concept that two external reductions do not contract redexes that have common residuals (although the contracted redexes may have 'inessential' common ancestors).

Definition 2.3 • Let $u \in U \subseteq t$ and $P : t \twoheadrightarrow o$. We call P *external* to U (resp. u) if P does not contract residuals of redexes in U (resp. residuals of u).

- Let $P : t_0 \stackrel{P_i}{\twoheadrightarrow} t_i \stackrel{u_i}{\rightarrow} t_{i+1} \twoheadrightarrow t_n$ and $Q : t_0 = s_0 \stackrel{Q_j}{\twoheadrightarrow} s_j \stackrel{v_j}{\rightarrow} s_{j+1} \twoheadrightarrow s_m$. Let $U_{i,j} = u_i/(Q_j/P_i)$ and $V_{i,j} = v_j/(P_i/Q_j)$ (see diagram). We call P *external* to Q if for any i, j, $U_{i,j} \cap V_{i,j} = \emptyset$.



Obviously, P is external to the set U iff it is external to any development of U, and is external to a redex u iff it is external to the reduction contracting u. Note that a reduction external to one complete development of U need not be external to all developments of U, and in general, externality is not invariant under $\approx_L$. For, consider a TRS $R = \{a \to a',\ f(x) \to b,\ g(x) \to c\}$, a term $t = f(g(a))$, and reductions $P : t \xrightarrow{a} f(g(a')) \xrightarrow{f} b$, $Q : t \xrightarrow{a} f(g(a')) \xrightarrow{g} f(c)$, and $N : t \xrightarrow{g} f(c)$. Then we have $Q \approx_L N$, P is external to N, but not to Q; and P is not external to $U = \{a, g(a)\}$.

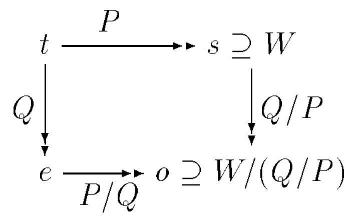
Recall from [19] that in concrete orthogonal rewrite systems, a redex is said to be *external* if its residuals never appear inside arguments of other redexes. It should be clear from the context to which concept of external we are referring.

Lemma 2.4 *Let $P : t_0 \xrightarrow{v_0} t_1 \xrightarrow{v_1} \ldots \to t_n$ be external to $U = \{u_1, \ldots, u_n\} \subseteq t_0$, and let $Q_0 : t_0 \twoheadrightarrow o_0$. Then $P' = P/Q_0$ is external to the set $U' = U/Q_0$.*

We conclude this section by presenting two fundamental lemmas, which generalize the [weak acyclicity] and [stability] axioms to many-step reductions [14].

Lemma 2.5 (Weak Acyclicity) *Let P, N be co-initial finite reductions in a DRS, and let P be external to N. Then $N \not\approx_L P$.*

Lemma 2.6 (Stability) *Let $P : t \twoheadrightarrow s$ be external to $Q : t \twoheadrightarrow e$, in a stable DRS, and let P create redexes $W \subseteq s$. Then the residuals $W/(Q/P)$ of redexes in W are created by P/Q, and Q/P is external to W.*



3 Construction of the $\trianglelefteq_L$-meet operation

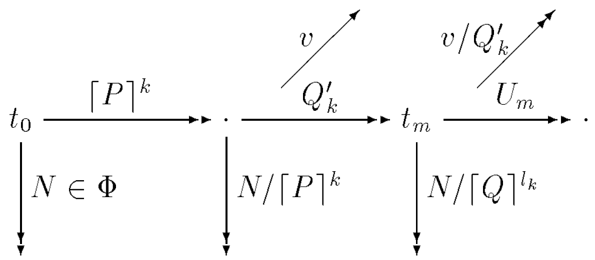
In this section we give a construction of the $\trianglelefteq_L$-meet operation (to be used in the subsequent sections) and prove its correctness. This implies that $\trianglelefteq_L$ is a complete lattice on Lévy-equivalence classes of reductions. This lattice was first constructed by Berry and Lévy [9], for the case of Recursive Program Schemes (RPSs), as the ideal completion of the upper semi-lattice on Lévy-equivalence classes of finite reductions.

Below, in our construction of $\trianglelefteq_L$-meets, Φ denotes a set of co-initial reductions in a DRS. Further, for example, we write $P \trianglelefteq_L \Phi$ iff $\forall Q \in \Phi.\ P \trianglelefteq_L Q$; $\lceil \Phi \rceil^k$ denotes the set of initial parts, of length k, of reductions in Φ; etc.

Definition 3.1 ($\trianglelefteq_L$-meet) Let Φ be a set of reductions starting from t, in a DRS. Then the $\trianglelefteq_L$-*meet* of reductions in Φ, written $\sqcap_L \Phi$, is defined as follows: Let $U \subseteq t$ be the maximal subset such that $U \trianglelefteq_L \Phi$, and let $t \overset{U}{\twoheadrightarrow} s$ be a complete U-development (or the multi-step contracting U). Then $\sqcap_L \Phi = U + \sqcap_L(\Phi/U)$.

Theorem 3.2 *Let Φ be a set of co-initial reductions in a DRS. Then $\sqcap_L \Phi$ is a (unique up to $\approx_L$) $\trianglelefteq_L$-meet of Φ.*

Proof. *Let $\sqcap_L \Phi = Q : t_0 \overset{U_0}{\twoheadrightarrow} t_1 \overset{U_1}{\twoheadrightarrow} t_2 \twoheadrightarrow \ldots$. It is immediate from the construction that $\sqcap_L \Phi \trianglelefteq_L \Phi$. Thus we need to show that for any $P : t_0 \twoheadrightarrow$ such that $P \trianglelefteq_L \Phi$, $P \trianglelefteq_L Q$, that is, for any $n < |P|$, $\lceil P \rceil^n \trianglelefteq_L Q$. We show this by induction on n. The case $n = 0$ (i.e., $P = \emptyset$) is clear. So let $n = k+1$, and let $\lceil P \rceil^k \trianglelefteq_L Q$. Then $\lceil P \rceil^k \trianglelefteq_L \lceil Q \rceil^{l_k}$ for some l_k. We can assume that $\lceil Q \rceil^{l_k}$ ends at t_m for some m. Thus $\lceil Q \rceil^{l_k} \approx_L \lceil P \rceil^k + Q'_k$ for some finite Q'_k, and $\Phi / \lceil Q \rceil^{l_k}$ consists of Q'_k-residuals of reductions in $\Phi / \lceil P \rceil^k$, up to $\approx_L$. Let v be the $(k+1)$th step of P. Since $P \trianglelefteq_L \Phi$, $v \trianglelefteq_L \Phi / \lceil P \rceil^k$, thus by the Cube Lemma, $(v/Q'_k) \trianglelefteq_L \Phi / \lceil Q \rceil^{l_k}$, and therefore $v/Q'_k \subseteq U_m$ by Definition 3.1. This means that $\lceil P \rceil^n \trianglelefteq_L Q$. Thus $P \trianglelefteq_L Q$, and $\sqcap_L \Phi$ is a $\trianglelefteq_L$-meet of Φ.*



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4 Stable sets of finite or infinite results

In this section, we will introduce the concepts of stability, regularity and superstability of sets of reductions, to cover usual concepts of infinite results, such as Böhm-trees. From this definition, our earlier concepts of stability and regularity for sets of finite results [14,15] can be obtained as a special case. Moreover, to any reduction P we will associate a stable set $\mathcal{S}_P$ of reductions, which will allow us to infer Discrete Normalization and Standardization results [26] from our normalization results in the next section.

We start by recalling the concepts of relative neededness, and of stability and regularity of sets of terms [14,15], and introduce superstability. The latter will allow us to extend our results concerning minimal relative normalization [15] to the abstract framework of SDRSs.

Definition 4.1 Let $\mathcal{S}$ be a set of terms in a DRS $\mathcal{R}$.

(1) We call a redex $u \in t$ *$\mathcal{S}$-needed* if at least one residual of it is contracted in any reduction from t to a term in $\mathcal{S}$, and call u *$\mathcal{S}$-unneeded* otherwise.

(2) We call $\mathcal{S}$ *stable* iff:
[RC] $\mathcal{S}$ is *closed under reduction*: if $t \in \mathcal{S}$ and $t \to s$, then $s \in \mathcal{S}$, and
[CUE] $\mathcal{S}$ is *closed under unneeded expansion*: for any $e \overset{u}{\to} o$ such that $e \notin \mathcal{S}$ and $o \in \mathcal{S}$, u is $\mathcal{S}$-needed.

(3) Further, we call a stable set $\mathcal{S}$ *regular* iff:
[Reg] In no term can an $\mathcal{S}$-unneeded redex duplicate an $\mathcal{S}$-needed one.

(4) Finally, we call a regular stable set $\mathcal{S}$ *superstable* iff:

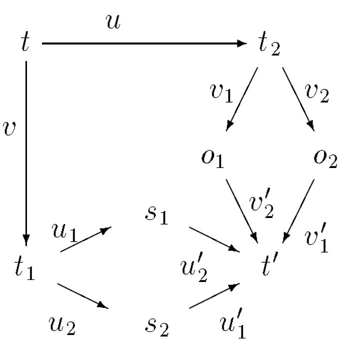
[Min] For any $\mathcal{S}$-normalizable term t, the set of all $\mathcal{S}$-normalizing reductions starting from t contains a unique, up to $\approx_L$, $\trianglelefteq_L$-minimal element. Such a reduction will be called *$\mathcal{S}$-minimal.*

Examples of stable sets are normal forms [19], head normal forms in the λ-calculus [6], weak head normal forms in the λ-calculus [1], constructor head normal forms for constructor TRSs [38], and root stable forms in orthogonal TRSs [37]. All the above sets are regular. Other examples include weak-head-normal forms (up to garbage-collection, modulo a congruence) in Yoshida's λf-calculus (an environment calculus) [46], the set of *answers* in the *call-by-need* λ-calculus of Ariola et al. [4], and flexible generalized head normal forms in the λ^v_{hd}-calculus of Xi [45]; all are conditional rewrite systems.

Although all normalizing reductions are minimal w.r.t. the set of normal forms, this is not so for other concepts of results. For example, for the set of head-normal form, only the reductions Lévy-equivalent to the leftmost-outermost head-normalizing reductions are minimal. Head-minimal reductions compute the principal head-normal forms [5], from which all other head-normal forms are accessible.

An example of an irregular stable set is given in [14]. The next example exhibits a regular stable set which is not superstable:

Example 4.2 Consider the SDRS given by the following diagram



where $u/v = \{u_1, u_2\}$, $v/u = \{v_1, v_2\}$, $v_1/v_2 = \{v'_1\}$, $v_2/v_1 = \{v'_2\}$, $u_1/u_2 = \{u'_1\}$, $u_2/u_1 = \{u'_2\}$, and let $\mathcal{S} = \{o_2, s_2, t'\}$. Then $\mathcal{S}$ is regular stable but not superstable, as one can easily verify that there is no $\mathcal{S}$-minimal reduction starting from t.

Definition 4.3 Given a set $\mathcal{S}$ of reductions in a DRS $\mathcal{R}$ and a term t in $\mathcal{R}$, we call t *$\mathcal{S}$-normalizable* if $\mathcal{S}$ contains a reduction P starting from t; P is then called $\mathcal{S}$-normalizing.

Definition 4.4 Let $\mathcal{S}$ be a set of reductions in a DRS.

(1) We call $u \in t$ *$\mathcal{S}$-unneeded* if there is a reduction $Q \in \mathcal{S}$ starting from t that is external to u, and call it *$\mathcal{S}$-needed* otherwise.

(2) Let P, Q be co-initial reductions. If P and Q are both finite, then we define $P \trianglelefteq_{\mathcal{S}} Q$ if P/Q is $\mathcal{S}$-unneeded. Otherwise, $P \trianglelefteq_{\mathcal{S}} Q$ if for any finite $P' \leq P$ there is a finite $Q' \leq Q$ such that $P' \trianglelefteq_{\mathcal{S}} Q'$. Further, $P \approx_{\mathcal{S}} Q$ iff

$P \trianglelefteq_{\mathcal{S}} Q$ and $Q \trianglelefteq_{\mathcal{S}} P$. We call $\trianglelefteq_{\mathcal{S}}$ and $\approx_{\mathcal{S}}$ respectively *$\mathcal{S}$-embedding* and *$\mathcal{S}$-equivalence*.

It is immediate from the definition that $\trianglelefteq_L \subseteq \trianglelefteq_{\mathcal{S}}$.

Lemma 4.5 *Let P, Q be co-initial reductions in a DRS. Then $P \trianglelefteq_L Q$ implies $P \trianglelefteq_{\mathcal{S}} Q$ and $P \approx_L Q$ implies $P \approx_{\mathcal{S}} Q$.*

Definition 4.6 Let $\mathcal{S}$ be a set of reductions in a DRS.

(1) We call $\mathcal{S}$ *stable* iff:
- **[CS]** $\mathcal{S}$ *is suffix-closed*: if $P' \notin \mathcal{S}$, then $P' + P'' \in \mathcal{S}$ implies $P'' \in \mathcal{S}$.
- **[CE]** $\mathcal{S}$ *is closed under $\mathcal{S}$-embedding*: $P \in \mathcal{S}$ and $P \trianglelefteq_{\mathcal{S}} Q$ implies $Q \in \mathcal{S}$.
- **[CN]** $\mathcal{S}$ *is closed under neededness*: every non-empty $P \in \mathcal{S}$ contracts at least one $\mathcal{S}$-needed redex.

(2) Furthermore, we call $\mathcal{S}$ *regular* iff:
- **[Reg]** In no term can an $\mathcal{S}$-unneeded redex duplicate an $\mathcal{S}$-needed one.

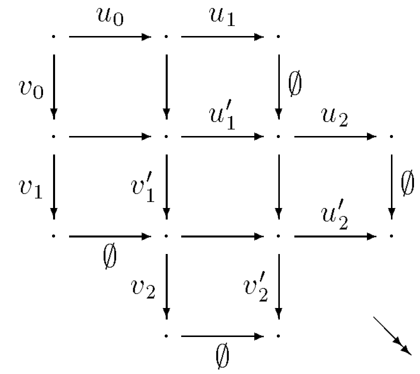
(3) Finally, we call $\mathcal{S}$ *superstable* iff:
- **[Min]** For any $\mathcal{S}$-normalizable term t, $\mathcal{S}$ contains a unique, up to $\approx_L$, $\trianglelefteq_L$-minimal element starting from t. Such reductions will be called $\mathcal{S}$-minimal.

Note that, for a set of terms $\mathcal{S}_{ter}$ in a DRS, stability of $\mathcal{S}_{ter}$ follows easily from stability of the *corresponding* set $\mathcal{S}_{red}$ of $\mathcal{S}_{ter}$-normalizing reductions – (possibly infinite) reductions starting outside $\mathcal{S}_{ter}$ and entering it. Indeed, [CE] (for $\mathcal{S}_{red}$) implies [RC] (for $\mathcal{S}_{ter}$). And [CN] implies [CUE] (since any step entering $\mathcal{S}_{ter}$ is an $\mathcal{S}_{red}$-normalizing reduction, thus is $\mathcal{S}_{red}$-needed and therefore $\mathcal{S}_{ter}$-needed too).

Conversely, if $\mathcal{S}_{ter}$ is stable, then $\mathcal{S}_{red}$ is stable as well: [CS] is immediate, [CN] follows from [CUE], and [CE] follows from [CUE] and [RC]. (Indeed, to prove [CE], let $P \in \mathcal{S}_{red}$ and $P \trianglelefteq_{\mathcal{S}_{red}} Q$. Further, let $P' \leq P$ be finite and $P' \in \mathcal{S}_{red}$. Then there is a finite $Q' \leq Q$ such that $P' \trianglelefteq_{\mathcal{S}_{red}} Q'$. i.e., P'/Q' is $\mathcal{S}_{red}$-unneeded, hence $\mathcal{S}_{ter}$-unneeded. By [RC], P'/Q' ends in $\mathcal{S}_{ter}$. And by [CUE], Q' too ends in $\mathcal{S}_{ter}$. Hence $Q \in \mathcal{S}_{red}$, and [CE] is proved.)

Now it is immediate that $\mathcal{S}_{ter}$ is regular, resp. superstable, iff so is $\mathcal{S}_{red}$. Furthermore, $\mathcal{S}$-neededness and related concepts are not affected if we view $\mathcal{S}_{ter}$ as the corresponding set $\mathcal{S}_{red}$ of reductions. Therefore, in the following, we will often tacitly identify $\mathcal{S}_{ter}$ with $\mathcal{S}_{red}$.

Given a reduction P in an SDRS $\mathcal{R}$, we can also construct its corresponding stable set $\mathcal{S}_P$ as follows. Let $\mathcal{R}_P$ be an SDRS whose terms are Lévy-equivalence classes $\langle Q \rangle_L$ of finite initial parts of reductions in $\langle P \rangle_L$, whose redexes are pairs $\langle Q \rangle_L \to \langle Q + u \rangle_L$, where u is a redex in the final term of Q, and the residual relation is the induced one: If $\langle Q \rangle_L \stackrel{u^*}{\to} \langle Q + u \rangle_L$ and $\langle Q \rangle_L \stackrel{v^*}{\to} \langle Q + v \rangle_L$ in $\mathcal{R}_P$, then for any $v' \in v/u$, $\langle Q + u \rangle_L \stackrel{v'^*}{\to} \langle Q + u + v' \rangle_L$ is a u^*-residual of v. (This definition is sound since $Q + u + v' \trianglelefteq_L P$.) $\mathcal{R}_P$ is an SDRS since its reduction graph is isomorphic to $\langle P \rangle_L$, and in addition, $\mathcal{R}_P$ is acyclic. Now $\mathcal{S}_P$ is defined as the set of all suffixes of reductions in $\mathcal{R}_P$ corresponding to reductions in $\langle P \rangle_L$ (hence reductions in $\mathcal{S}_P$ are 'maximal' in some sense). Hence $\mathcal{S}_P$ is suffix-closed, and it satisfies [CE] since all co-initial reductions in $\mathcal{R}_P$ are Lévy-equivalent (because of the above 'maximality' property). When P is finite, the last step of any (non-empty) reduction in $\mathcal{S}_P$ is $\mathcal{S}_P$-needed by Lemma 2.5, implying [CN] and thus stability of $\mathcal{S}_P$. When P is infinite, [CN] does not hold in general (in SDRSs), as can be seen from the diagram below (only a finite part of the infinite reduction graph is shown). Each non-empty horizontal step creates its next horizontal step, and similarly for vertical steps; every redex has at most one residual on the opposite side of each elementary diagram, and erasure is indicated by empty steps $\emptyset$. The SDRS axioms are not violated. Further, the infinite reductions $u_0 + u_1 + \dots$ and $v_0 + v_1 + \dots$ are Lévy-equivalent but they contract different, hence unneeded, redexes. Clearly, no normalization-by-neededness theory can be developed for such reductions.



Let us recall concepts related to neededness w.r.t. particular reductions P, or discrete neededness, from [26].

Definition 4.7 • Let $P : t \twoheadrightarrow$ and $u \in t$, in a DRS. We call u *P-needed* if there is no $Q \approx_L P$ that is external to u, and call it *P-unneeded* otherwise.

• Let $Q : t \twoheadrightarrow$, $P : t \overset{P'}{\twoheadrightarrow} s \twoheadrightarrow$, and $u \in s$, in a DRS. We say that u is Q-needed, or more precisely, $P'u$ is Q-needed, if u is Q/P'-needed. We call P *Q-needed* if every redex contracted in P is Q-needed. We call P *self-needed* or *standard* if it is P-needed.

It is easy to verify that P-neededness of redexes occurring in terms of reductions in $\langle P \rangle_L$ coincides with $\mathcal{S}_P$-neededness, and in particular standard reductions P are simply $\mathcal{S}_P$-needed reductions. Hence the stability concept for sets of reductions allows to unify the existing relative and discrete normalization theories (as far as normalization-by-neededness is concerned).

It is easy to check that for example the sets of reductions 'computing' Böhm-trees [5] or Lévy-Longo-trees [30,33] or Berarducci-trees [7] are stable and regular. To make this precise, let us consider the Böhm-tree semantics of the λ-calculus. Recall that the *immediate syntactic value* $\omega(t)$ of a λ-term t is defined by: $\omega(t) = \bot_{\mathcal{B}}$ if t is not a head normal form, and $\omega(t) = \lambda x_1 \dots x_m . y\omega(t_1) \dots \omega(t_n)$ if $t = \lambda x_1 \dots x_m . y t_1 \dots t_n$, where $\bot_{\mathcal{B}}$ is a special constant which denotes the 'undefined'. Then the *Böhm-tree* or *Böhm-normal-*

form $BT(t)$ of t, which is the *value* of t according to *Böhm-semantics*, is defined by $BT(t) = \trianglelefteq_\Omega\!-lub\{\omega(t) \,|\, t \twoheadrightarrow_\beta s\}$, where $\trianglelefteq_\Omega$ is the minimal context-closed ordering on λ-$\bot_\mathcal{B}$-terms (i.e., λ-terms possibly containing occurrences of $\bot_\mathcal{B}$) generated by $\bot_\mathcal{B} \trianglelefteq_\Omega t$ for any λ-$\bot_\mathcal{B}$-term t.

Now let us define the *Böhm-approximant computed by* $P : t_0 \to t_1 \to \ldots$, written $\omega(P)$, by $\omega(P) = \trianglelefteq_\Omega - \, lub\{\omega(t_i) \,|, i = 0, 1 \ldots .\}$. It is well known that $t \twoheadrightarrow s$ implies $\omega(t) \trianglelefteq_\Omega \omega(s)$, thus in particular for a finite $P : t_0 \twoheadrightarrow t_k$, $\omega(P) = \omega(t_k)$. Now we can define that $P : t_0 \twoheadrightarrow$ *computes* $BT(t_0)$ iff $\omega(P) = BT(t_0)$. Let us call a subterm of t an $\bot_\mathcal{B}$-subterm if its corresponding occurrence in $\omega(t)$ is $\bot_\mathcal{B}$. Recall that the standard way to approximate $BT(t_0)$ sequentially is to construct 'iterated head-reductions' $P : t_0 \overset{P_0}{\twoheadrightarrow} t_1 \overset{P_1}{\twoheadrightarrow} \ldots$, where P_i is the head-reduction computing the principal head normal form of one of the $\bot_\mathcal{B}$-subterms of t_i having a head normal form. In order to compute $BT(t_0)$, P must be 'fair', i.e., every $\bot_\mathcal{B}$ subterm of every t_i having a head normal form must eventually be dealt with in P; we call such reductions P *canonical Böhm-reductions.*

Regularity and stability of the set $\mathcal{S}_\mathcal{B}$ of reductions computing Böhm-trees follow easily from the top-down manner of construction of Böhm-trees: [CS] is immediate; [CN] follows from the fact that head redexes of $\bot_\mathcal{B}$-subterms of any λ-term t are $\mathcal{S}_\mathcal{B}$-needed; and [CE] follows from a simple observation that $Q \in \mathcal{S}_\mathcal{B}$ iff $P \trianglelefteq_{\mathcal{S}_\mathcal{B}} Q$ for some canonical Böhm-reduction P. The axiom [Reg] follows from the fact that if a β-redex $u \in t$ is nested in an $\mathcal{S}_\mathcal{B}$-unneeded β-redex v, then for any canonical Böhm-reduction $P : t \twoheadrightarrow$, P does not contract residuals of v, and therefore it does not contract residuals of u as any residual of u is nested inside a residual of v. Finally, [Min] follows from the fact that $P \trianglelefteq_L \sqcap_L \mathcal{S}_\mathcal{B}^*$ for any canonical Böhm-reduction P, where $\mathcal{S}_\mathcal{B}^*$ is the subset of reductions in $\mathcal{S}_\mathcal{B}$ that are co-initial with P (note that, since $P \in \mathcal{S}_\mathcal{B}$, $P \trianglelefteq_L \sqcap_L \mathcal{S}_\mathcal{B}^*$ implies $P \approx_L \sqcap_L \mathcal{S}_\mathcal{B}^*$). To prove $P \trianglelefteq_L \sqcap_L \mathcal{S}_\mathcal{B}^*$, assume $P : t = t_0 \overset{u_0}{\to} t_1 \overset{u_1}{\to} \ldots$ and $\sqcap_L \mathcal{S}_\mathcal{B} : t = s_0 \overset{U_0}{\twoheadrightarrow} s_1 \overset{U_1}{\twoheadrightarrow} \ldots$. Since u_0 is external, it must be contracted by any reduction in $\mathcal{S}_\mathcal{B}$, hence $u_0 \in U_0$ and hence $u_0/U_0 = \emptyset$. Further, P/U_0 remains external, and we can prove similarly that the first contracted redex in P/U_0 must belong to U_1, and so on. This means that, $P/ \sqcap_L \mathcal{S}_\mathcal{B} = \emptyset$, that is, $P \trianglelefteq_L \mathcal{S}_\mathcal{B}$.

Note that $N \trianglelefteq_{\mathcal{S}_\mathcal{B}} Q$ implies $\omega(N) \trianglelefteq_\Omega \omega(Q)$, but $N \triangleleft_{\mathcal{S}_\mathcal{B}} Q$ need not imply $\omega(N) \triangleleft_\Omega \omega(Q)$. This is because the ordering $\trianglelefteq_{\mathcal{S}_\mathcal{B}}$ is more subtle than $\trianglelefteq_\Omega$ and can take into account (or observe) parts of computations that occur in the $\bot_\mathcal{B}$-subterms, while $\omega(t) = \bot_\mathcal{B}$ for any t not in head normal form, irrespectively of 'how far' is t from being a head normal form. For example, let $t = Ix(Ix)$ and $N : t \overset{I}{\to} (Ix)x$; then $\emptyset \triangleleft_{\mathcal{S}_\mathcal{B}} N$ but $\omega(\emptyset) = \omega(N) = \bot_\mathcal{B}$. Thus $\trianglelefteq_{\mathcal{S}_\mathcal{B}}$, defined on reductions rather than on terms, can be seen as a refinement of the classical Böhm topology on trees [5].

Thus stability of sets of reductions is a natural generalization of stability for sets of finite results. It is also consistent with the definition of stability of sets of configurations in prime event structures with erasure in [27,28].

5 Relative Normalization in SDRSs

In this section, we will generalize the Relative Hypernormalization theorem to regular stable sets of finite or infinite reductions, in stable DRSs $\mathcal{R}$. We then show how $\mathcal{S}$-minimal reductions can be constructed in $\mathcal{R}$.

We begin by showing that, for finite or infinite results $\mathcal{S}$, $\mathcal{S}$-unneeded redexes cannot create $\mathcal{S}$-needed ones, and that residuals of $\mathcal{S}$-unneeded redexes remain unneeded. When $\mathcal{S}$ is regular, this enables us to construct an $\mathcal{S}$-needed variant of any $\mathcal{S}$-normalizing reduction, in any SDRS.

Lemma 5.1 *Let $\mathcal{S}$ be a set of reductions satisfying [CE] and [CS], in an SDRS (in particular, $\mathcal{S}$ may be regular), and let $P : t \overset{u}{\to} s \twoheadrightarrow \ \in \mathcal{S}$. Further:*

(1) *Let v' be a u-residual of $v \in t$, and let v be $\mathcal{S}$-unneeded. Then so is v'.*

(2) *Let u create $v \in s$, and let u be $\mathcal{S}$-unneeded. Then so is v.*

Proof.

(1) *Since v is $\mathcal{S}$-unneeded, there is $Q : t \twoheadrightarrow \ \in \mathcal{S}$ that is external to v. Then Q/u is external to v' by Lemma 2.4, and $Q/u \in \mathcal{S}$ by [CE], Lemma 4.5 and [CS] (indeed, $Q \in \mathcal{S} \wedge Q \trianglelefteq_L u + Q/u$ implies $u + Q/u \in \mathcal{S}$ implies $Q/u \in \mathcal{S}$). Hence v' is $\mathcal{S}$-unneeded.*

(2) *Since u is $\mathcal{S}$-unneeded, there is $Q : t \twoheadrightarrow \ \in \mathcal{S}$ that is external to u. By Lemma 2.6, Q/u is external to v, and $Q/u \in \mathcal{S}$ by [CE] and [CS] (see the proof of (1)). Thus v is $\mathcal{S}$-unneeded.*

□

Definition 5.2 Let $\mathcal{S}$ be a set of reductions in an SDRS $\mathcal{R}$.

(1) We call P *$\mathcal{S}$-(un)needed* if it contracts only $\mathcal{S}$-(un)needed redexes. We call P *$\mathcal{S}$-quasi-needed* if it contracts infinitely many $\mathcal{S}$-needed redexes.

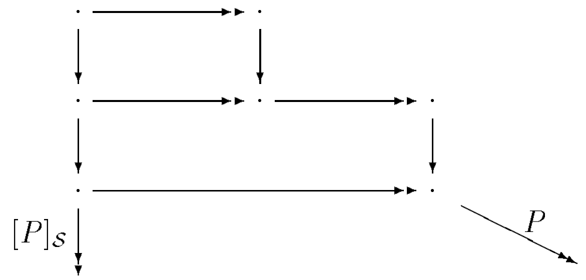
(2) The *$\mathcal{S}$-needed part of $P : t \twoheadrightarrow$* , written $[P]_\mathcal{S}$, is a finite or infinite reduction defined by: $[P]_\mathcal{S} = u + [P/u]_\mathcal{S}$, where $u \in t$ is the redex whose residual along P is contracted first among $\mathcal{S}$-needed steps in P, if any.

(3) We call $P : t_0 \to t_1 \to \ldots$ *$\mathcal{S}$-needed fair* if for any $\mathcal{S}$-needed redex $v \in t_i$, $v_i \trianglelefteq_\mathcal{S} \lceil P \rceil_i$.

It follows immediately from Lemma 5.1 that $[P]_\mathcal{S}$ is $\mathcal{S}$-needed. Note that $[P]_\mathcal{S}$ is not defined uniquely as its $(k+1)$th step depends on the particular sequentializations of multi-steps (i.e., complete developments of redex sets) in $P/\lceil [P]_\mathcal{S} \rceil^k$, but this does not cause any difficulties.

Lemma 5.3 *Let P be a finite or infinite reduction in an SDRS $\mathcal{R}$, and let $\mathcal{S}$ be a regular stable set of reductions in $\mathcal{R}$. Then $[P]_\mathcal{S}$ is an $\mathcal{S}$-needed reduction whose length coincides with the number of $\mathcal{S}$-needed steps in P, and $P \approx_\mathcal{S} [P]_\mathcal{S}$.*

Proof. *Immediate from the fact that, in the notation of Definition 5.2, u has at most one residual along P (by the regularity of $\mathcal{S}$), and one of the residuals of u is contracted in P (see the diagram, where all horizontal steps* *are $\mathcal{S}$-unneeded and all vertical ones are $\mathcal{S}$-needed).*



□

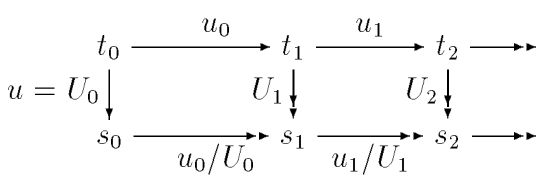
Next we show that, unless it is contracted, an $\mathcal{S}$-needed redexes has at least one $\mathcal{S}$-needed residual. Therefore, residuals of $\mathcal{S}$-quasi-needed reductions remain so. It follows that an $\mathcal{S}$-normalizable term t cannot possess an $\mathcal{S}$-quasi-needed reduction if at least one $\mathcal{S}$-normalizing reduction starting from t is finite.

Lemma 5.4 *Let $\mathcal{S}$ be a regular stable set of reductions in an SDRS $\mathcal{R}$, and let $P : t \overset{u}{\to} s \overset{P'}{\twoheadrightarrow} \in \mathcal{S}$. Further, let $u \neq v \in t$, and let v be $\mathcal{S}$-needed. Then v has at least one $\mathcal{S}$-needed residual in s.*

Proof. *Since $P \in \mathcal{S}$ and v is $\mathcal{S}$-needed, $v/u = V \neq \emptyset$. Suppose on the contrary that none of the redexes in V is $\mathcal{S}$-needed. By [CS] and Lemma 5.3, $[P']_\mathcal{S} \in \mathcal{S}$, and since $[P']_\mathcal{S}$ is $\mathcal{S}$-needed, it is external to V by Lemma 5.1.(1). Hence $u + [P']_\mathcal{S}$ is external to v, and $u + [P']_\mathcal{S} \in \mathcal{S}$ by [CE], contradicting $\mathcal{S}$-neededness of v.* □

Lemma 5.5 *Let $\mathcal{S}$ be a regular stable set of reductions in an SDRS, let t_0 have an $\mathcal{S}$-quasi-needed reduction, and let $t_0 \overset{u}{\to} s_0$. Then s_0 also has an $\mathcal{S}$-quasi-needed reduction.*

Proof. *By Lemma 5.3, t_0 has an infinite $\mathcal{S}$-needed reduction $P : t_0 \overset{u_0}{\to} t_1 \overset{u_1}{\to} \ldots$. Let $U_i = u/(u_0 + \ldots + u_{i-1})$, $i = 0, 1, \ldots$ (see the diagram below). It follows from finiteness of developments that there are infinitely many numbers k such that $u_k \notin U_k$ (otherwise there should be a number m such that $t_m \overset{u_m}{\to} t_{m+1} \overset{u_{m+1}}{\to} \ldots$ is an infinite U_m-development). By Lemma 5.4, u_k has at least one $\mathcal{S}$-needed U_k-residual in s_k, i.e. u_k/U_k contains at least one $\mathcal{S}$-needed step. Hence P/u is $\mathcal{S}$-quasi-needed.*



□

The following theorem justifies the stability concept for sets of reductions:

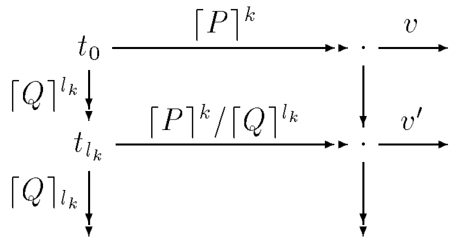
Theorem 5.6 *Let $\mathcal{S}$ be a regular stable set of reductions in an SDRS $\mathcal{R}$, and let t be $\mathcal{S}$-normalizable.*

(1) *Any $\mathcal{S}$-needed fair reduction starting from t is $\mathcal{S}$-normalizing.*

(2) *If $\mathcal{S}$ contains a finite reduction starting from t, then t does not possess a reduction in which $\mathcal{S}$-needed redexes are contracted infinitely many times.*

Proof.

(1) *By Lemma 5.3, there is an $\mathcal{S}$-needed reduction $P : t \twoheadrightarrow\ \in \mathcal{S}$. Let Q be $\mathcal{S}$-needed fair. By [CE], it is enough to prove that $P \trianglelefteq_{\mathcal{S}} Q$, which requires proving that $\lceil P \rceil^k \trianglelefteq_{\mathcal{S}} Q$ for every finite initial part $\lceil P \rceil^k$ of P. Suppose $\lceil P \rceil^k \trianglelefteq_{\mathcal{S}} Q$, and let us prove $\lceil P \rceil^{k+1} \trianglelefteq_{\mathcal{S}} Q$. Then $\lceil P \rceil^k / \lceil Q \rceil^{l_k}$ is $\mathcal{S}$-unneeded for some l_k. Let v be the $(k+1)$th step of P. If v has an $\mathcal{S}$-needed $\lceil Q \rceil^{l_k} / \lceil P \rceil^k$-residual v', then by Lemma 5.1 v' must be a residual of some $\mathcal{S}$-needed redex v'' in the final term t_{l_k} of Q_{l_k} (see the diagram), and $v'' \trianglelefteq_{\mathcal{S}} \lceil Q \rceil_k$ since Q is $\mathcal{S}$-needed fair. Thus, by the Cube Lemma and Lemma 5.1, $v' \trianglelefteq_{\mathcal{S}} \lceil Q \rceil_{l_k} / (\lceil P \rceil^k / \lceil Q \rceil^{l_k})$, and we are done.*



(2) *Let $P : t \twoheadrightarrow s$ and $P \in \mathcal{S}$. Suppose on the contrary that there is an $\mathcal{S}$-quasi-needed Q starting from t. Then by Lemma 5.5 Q/P is $\mathcal{S}$-quasi-needed as well – a contradiction, since $P \in \mathcal{S}$ implies that Q/P is $\mathcal{S}$-unneeded.*

□

Based on the result in section 3, we will now give a sufficient and necessary condition for superstability of regular stable sets of finite results.

Definition 5.7 Let $\mathcal{S}$ be a set of reductions in a DRS.

(1) We call $u \in t$ *$\mathcal{S}$-erased* if u does not have a residual under any $\mathcal{S}$-normalizing reduction.

(2) We call a reduction *$\mathcal{S}$-erased* if it only contracts $\mathcal{S}$-erased redexes.

Clearly, a reduction P is $\mathcal{S}$-erased iff $P \trianglelefteq_L Q$ for every $Q \in \mathcal{S}$. Note that $\mathcal{S}$-erased redexes need not be $\mathcal{S}$-needed (e.g., when $\mathcal{S}$ is the set of normal forms in an orthogonal TRS that has an erasing rule, say $f(x) \to a$). However, any reducible normalizable term t in an orthogonal (first or higher-order) TRS has an *external* redex [19]. And such redexes are both erased and needed w.r.t. to the regular stable sets considered in the literature.

Theorem 5.8 *Let $\mathcal{S}$ be a regular stable set of terms, in an SDRS. Then $\mathcal{S}$ is superstable (i.s, satisfies [Min]) iff any $\mathcal{S}$-normalizable term $t \notin \mathcal{S}$ contains an $\mathcal{S}$-erased $\mathcal{S}$-needed redex.*

Proof. *Let $\mathcal{S}_{red}$ be the set of all $\mathcal{S}$-normalizing reductions starting from t, and let $\sqcap_L \mathcal{S}_{red} : t = t_0 \stackrel{U_0}{\twoheadrightarrow} t_1 \stackrel{U_1}{\twoheadrightarrow} \ldots$.*

($\Leftarrow$) By the assumption and the construction of $\sqcap_L \mathcal{S}_{red}$, every multi-step of $\sqcap_L \mathcal{S}_{red}$ contracts an $\mathcal{S}$-needed redex. Hence $\sqcap_L \mathcal{S}_{red} \in \mathcal{S}_{red}$ by Theorem 5.6 and $\sqcap_L \mathcal{S}_{red}$ is $\mathcal{S}$-minimal by Theorem 3.2, which imply [Min].

($\Rightarrow$) Since, by [Min], $\sqcap_L \mathcal{S}_{red}$ is $\mathcal{S}$-normalizing, it contracts a $\mathcal{S}$-needed redex by [CUE]. Let i be the smallest number such that U_i contains an $\mathcal{S}$-needed redex, say v_i. Suppose on the contrary that $i \neq 0$. Then, by Lemma 5.1, t_{i-1} contains a $\mathcal{S}$-needed redex v_{i-1} for which v_i is the unique residual (since by the minimality of i, all redexes in U_{i-1} are $\mathcal{S}$-unneeded). But v_{i-1} is erased along $\sqcap_L \mathcal{S}_{red}$ since v_i is, and by the $\mathcal{S}$-minimality of $\sqcap_L \mathcal{S}_{red}$, v_{i-1} is $\mathcal{S}$-erased. Hence we must have $v_{i-1} \in U_{i-1}$ – a contradiction. □

This theorem gives a more 'local' definition of superstability for regular stable sets of finite results. By combining this with the Relative Hypernormalization theorem, we get the following corollary:

Corollary 5.9 *(***Minimal Relative Normalization***) Let $\mathcal{S}$ be a superstable set of terms in an SDRS, and let t be an $\mathcal{S}$-normalizable term not in $\mathcal{S}$. $\mathcal{S}$-minimal $\mathcal{S}$-normalizing reductions arise from repeatedly contracting $\mathcal{S}$-needed $\mathcal{S}$-erased redexes in t. A finite number of $\mathcal{S}$-unneeded but $\mathcal{S}$-erased redexes may also be contracted without losing $\mathcal{S}$-minimality.*

6 Concluding remarks

We have formalized a concept of 'infinite results', and proved a normalization theorem relative to such results, in an abstract framework of SDRSs. We also studied minimal relative normalization in SDRSs. These results unify and extend several earlier results on relative and discrete normalization. Furthermore, concepts of stable sets of results and corresponding orderings introduced in this paper allow SDRSs to be viewed as semi-distributive domains, or 'domains with duplication' [25].

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