An Abstract Böhm Normalization

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Overview

- Normalization by neededness
- Relative normalization
- Abstract relative normalization
- Infinite results of finite terms
- Stability, regularity and superstability
- Böhm normalization and minimality results
- Conclusions

Normalization by neededness

- Developed by Huet & Lévy, 1979.
- In an orthogonal TRS, a redex in term t is *needed* if its residual is constructed in every normalizing reduction of t.
- If t has a normal form, it can be found by repeatedly contracting needed redexes

Normalization wrt other sets of normal forms

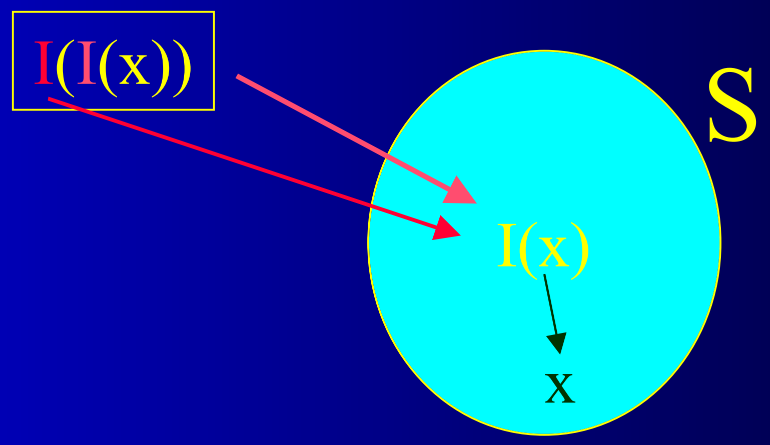
- Barendregt et al. studied normalization wrt *head-normal* forms, in the λ–calculus
- Maranget - *weak-head-normal* forms
- Nöcker – *constructor head-normal* forms
- Middeldorp – *root-stable* forms
- Glauert & Khasidashvili formalized a well-behaved concept of ‘partial results’

Relative neededness & Stable sets of results

- Let S be a set of terms. A redex u in term t is S-*needed* if any S-normalizing reduction starting from t contracts a residual of u.
- A set S of terms is stable if:
  - It is closed under reduction
  - Every step entering S is S-needed

A non-stable set

- A non-stable set: Set S={I(x),x}
- Reduction relation: R ={I(x) → x}
- Term I(I(x)) has no S-needed redex; it has an S-normal form (actually, it has two).



Normalization wrt a reduction

- Let P: t→→…→s be a reduction
- A redex u in t is P-needed if it is contracted in any reduction Lévy-equivalent to P.
- P is self-needed or standard if it only contracts P-needed redexes.
- For '*regular*' reductions P, we have shown that by contracting P-needed P-erased redexes we can built standard reductions Lévy-equivalent to P.

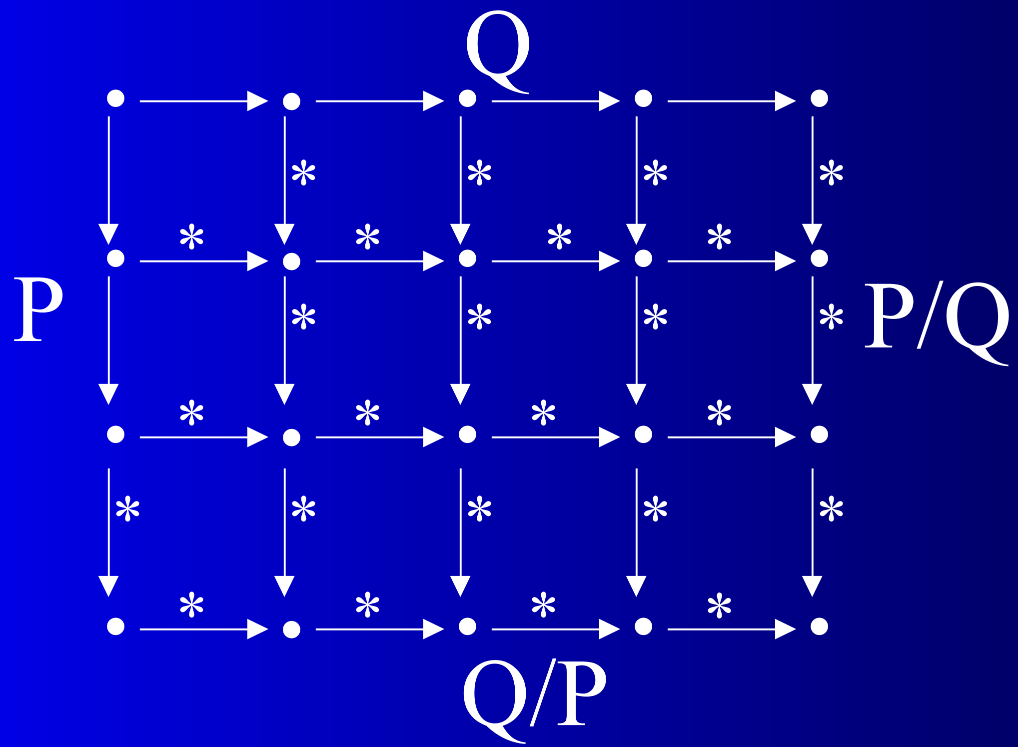
Lévy-equivalence

- *Lévy*-equivalence on finite co-initial reductions is generated by the axioms:
  - $U+V/U =_L V+U/V$
  - $P=P' \Rightarrow N+P+Q =_L N+P'+Q$

  where U (resp. V) denotes *complete development* of redex set U, and U/V denotes the set of residuals of redexes in U after performing V, as well as the corresponding complete development.

Lévy-equivalence (cont.)

- Klop diagram: $P =_L Q$ ➔ P/Q=Q/P=0



Deterministic Residual Structures (DRS)

- A ***DRS*** consists of
  - An *Abstract Reduction System* *A=(Ter,Red,→)*
    - ***Ter*** is set of objects, called *terms*
    - ***Red*** is set of *redexes* (or *redex occurrences*)
    - → associates to every redex its source and target terms; redexes are written u : t → s
    - A term may have a *finite* number of redexes
  - A *residual relation*, denoted /, between redexes in the source and target terms of →.

Deterministic Residual Structures (cont.)

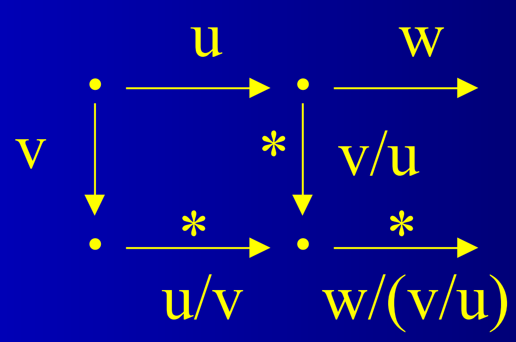
- / satisfies three ‘*permutation*’ axioms:
  - If t → s, then a redex u in s may be a residual of at most one redex v in t; otherwise u is *created*;
  - u/u = 0 (empty reduction);
  - All developments *terminate*; all complete developments of a set U of redexes in t end at the *same* term; and residuals of a redex v in t under all complete developments of U are the *same*.

Deterministic Residual Structures (cont.)

- The residual relation extends to finite reductions by transitivity, and Lévy equivalence can be defined.
- / satisfies an '*advanced*' axiom:
  - [weak acyclicity] (E.Stark [Sta89])
    - u/v=0 & !u=v → !v/u=0
    - u/v=0 & u≠v → v/u≠0

Stable DRSs

- A DRS is *stable* if in addition the following axiom is satisfied:
  - [stability] (modification of [GLM'92])
    - !u=v & u *creates* w → v/u *creates* w/(v/u)
    - u≠v & u *creates* w → v/u *creates* w/(v/u)



$\leqslant_L$ -embedding

- Define P $\leqslant_L$ Q iff P/Q=0.
- **Theorem**: Let Φ be a set of reductions starting from t. Then $\leqslant_L$-meet $\sqcap_L \Phi$ of reductions in Φ can be computed as follows:

  $-\sqcap_L \Phi = U + \sqcap_L (\Phi/U)$

  where U is the set of all redexes t such that:

  $U \leqslant_L \Phi$

Relative neededness

- Let S be a set of reductions in a DRS
- We call a redex u in t S-*unneeded* if there is a Q in S that is external to u (i.e., does not contract residuals of u), and is S-*needed* otherwise.
- A reduction with S-(un)needed steps is S-*(un)needed*

Stable ordering: $\leqslant_s$

- Let S be a set of reductions in a DRS
- If P and Q are finite, define $P \leqslant_s Q$ iff P/Q is S-unneeded.
- If P and/or Q are infinite, define $P \leqslant_s Q$ iff for any initial part P' of P there is an initial part Q' of Q such that $P' \leqslant_s Q'$.
- S-*equivalence*: $P =_S Q$ iff $P \leqslant_s Q$ & $Q \leqslant_s P$.

Stable sets of reductions

- A set S of reductions is called *stable* iff:
  - !P' in S & P'+P" in S ➔ P" in S
  - P is S & $P \leqslant_S Q$ ➔ Q in S
  - Every non-empty P in S contracts at least one S-needed redex

Regular and superstable sets

- A set S of reductions is called *regular* iff:
  - In no term can an S-unneeded redex duplicate and S-needed redex
- S is called *superstable* iff:
  - For any S-normalizable term t, S contains a unique, up to $=_L$, $\leqslant_L$-minimal reduction starting from t. Such reductions are called S-*minimal*.

Böhm Normalization

- P : $t_0 \rightarrow t_1 \rightarrow \dots$ is *S-needed fair* if for any S-needed redex v_i in t_i, $v_i \leqslant_S P_i$, where P_i is the suffix of P starting from t_i.
- A redex u in t is *S-erased* if it does not have a residual under any S-normalizing reduction starting from t.

Böhm Normalization (cont.)

- **Theorem**: Let S be a regular set of reductions, and let t be S-normalizable.
  - Any S-needed fair reduction starting from t is S-normalizing.
  - If S contains a finite reduction starting from t, then t does not have a reduction in which infinitely many times S-needed redexes are contracted.

$\leqslant_S$

A characterization of superstability

- **Theorem**: Let S be a regular stable set of terms, in an SDRS. Then S is superstable iff any S-normalizable term t not in S contains an S-erased S-needed redex.

Minimal relative normalization

- **Theorem**: Let S be a superstable set of terms in an SDRS, and let T be S-normalizable term not in S. S-minimal S-normalizing reductions arise from repeatedly contracting S-needed S-erased redexes. A finite number of S-unneeded but S-erased redexes may also be contracted without loosing S-minimality.

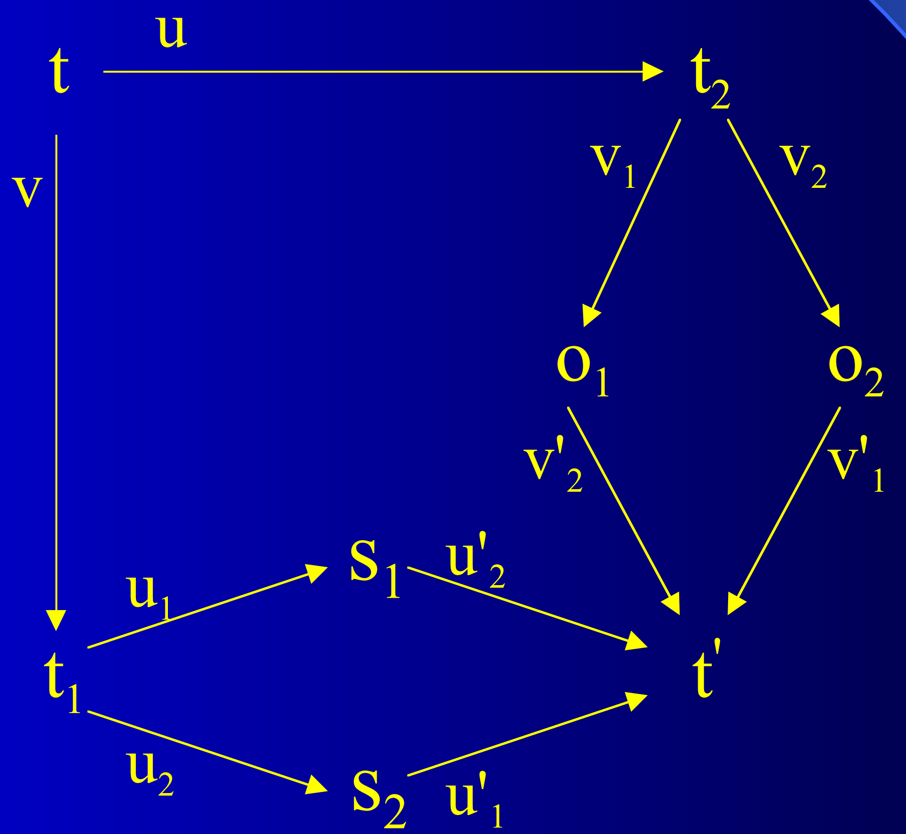
Related work

- On Abstract reductions with residuals:
  - Started by Stark, and by Gonthier, Lévy and Mellies; further developed by Melliès.
- On minimal reductions:
  - Our early minimality results (for orthogonal ERSs) were inspired by Maranget.
  - Melliès later proved a minimality result by using nesting axioms.

Conclusions

- We have unified our earlier relative and discrete normalization results.
- We use this abstract framework in a number of articles to study operational, denotational as well as event based semantics of orthogonal rewrite systems.
- Generalizing these results to infinite terms (with infinitely many redexes) requires enriching SDRSs with axioms studied by Kennaway (CWI report).

Example 4.2



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