Dynamics of Iain M. Banks' Orbitals

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Note

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1 The problem

An Orbital consists of a ribbon of matter formed into a ring spinning in its own plane, while orbiting a star. The inner surface is inhabited, centrifugal force substituting for gravity.

If the ring plane is inclined to the orbital plane, the star exerts a torque on the ring, causing it to precess. What effects will this have? Under what conditions is the precession of sufficient magnitude to keep the ring in a constant attitude towards the star?

2 Definitions

2.1 Constants defining the system

m	mass of Orbital
r	radius of Orbital
ω	angular velocity of Orbital about its centre
θ	tilt of Orbital
M	mass of star
R	radius of orbit
Ω	angular velocity of Orbital about the star
G	gravitational acceleration due to the star at the distance of the Orbital

2.2 Derived quantities

g centrifugal acceleration experienced by inhabitants of the Orbital
T tidal torque exerted by the star on the Orbital
J angular momentum of the Orbital about its centre
ψ precessional angular velocity
y number of days in the Orbital's year

3 Symbolic calculation

$$
\begin{aligned}
g &= r\omega^2 && (1)\\
G &= R\Omega^2 && (2)\\
J &= mr^2\omega && (3)\\
T &= \int_{\alpha=0}^{2\pi} r\sin\alpha\sin\theta\,\frac{m\,d\alpha}{2\pi}\,\frac{2G}{R}\,r\sin\alpha\cos\theta && (4)\\
&= \frac{Gmr^2}{R}\sin\theta\cos\theta && (5)\\
&= mr^2\Omega^2\sin\theta\cos\theta && (6)\\
\psi &= \frac{T}{J\sin\theta} = \frac{G\cos\theta}{R\omega} = \frac{\Omega^2\cos\theta}{\omega} && (7)\\
y &= \omega/\Omega && (8)
\end{aligned}
$$

The above expression for T is for the case where the ring plane and the orbital plane intersect along a line tangent to the orbit. We consider other configurations later. The components of the integral are:

$r\sin\alpha\sin\theta$ displacement of mass element from orbital plane
$m\,d\alpha/2\pi$ mass element
$2G/R$ tide
$r\sin\alpha\cos\theta$ displacement of mass element along orbital radius

For one precession per year, we must have $\psi = \Omega$, from which we derive:

$$\omega = \Omega\cos\theta \qquad (9)$$

In other words, $y = \cos\theta$, and the year is less than a day long. However, the tide in such a case would be strong enough to distort the ring into a straight line (assuming it has effectively zero bending resistance). We conclude that a precession-locked orbit is not possible.

4 Numerical calculation

We shall try to duplicate Earth living conditions.

$$
\begin{aligned}
g &= 9.81\,m/s^2 && (10)\\
\omega &= 2\pi/86400 = 7.27\cdot10^{-5}\,s^{-1} && (11)\\
\theta &= 23\,\mathrm{deg} = 0.401\,\mathrm{rad} && (12)
\end{aligned}
$$

The choice of θ means that everyone on the Orbital experiences the equivalent exposure to the sun as people in the tropics on Earth.

This gives us the radius of the Orbital:

$$r = g/\omega^2 = 1.89 \cdot 10^9\, m \tag{13}$$

This is about 5 times the distance from the Earth to the Moon.

Assume an Earth-like star and Earth-like orbit:

$$\begin{aligned} R &= 1.49 \cdot 10^{11}\, m && (14) \\ \Omega &= 2\pi/31000000 = 2.02 \cdot 10^{-7}\, s^{-1} && (15) \end{aligned}$$

Then the precession rate is

$$\begin{aligned} \psi &= \frac{\Omega^2 \cos\theta}{\omega} = 5.14 \cdot 10^{-10}\, s^{-1} && (16) \\ \Omega/\psi &= 396 && (17) \end{aligned}$$

So the precession of the angular momentum, if it were to be constant, would rotate it once every four centuries. But it is not constant, as we shall see later.

5 The universal night

As the precession cannot be synchronised to the orbit, the planet will experience seasons alternating between summer and some sort of colder season, depending on the axial tilt, twice per orbit. In addition, when the ring is edge-on to the star at the peak of summer, the entire inner surface is in shadow, one half being eclipsed by the other. How long will the eclipse last?

Let the transverse width of the ring be w. To an inhabitant on the oppposite side of the ring from the sun, the sun will be eclipsed for the time it takes the ring to traverse an angle of its orbit equal to that subtended by a width $w/\sin\theta$ at a distance of $2r$. This time is $\frac{w}{2r\Omega\sin\theta}$ seconds.

For Earth-like conditions, this comes to 3460 seconds (just under an hour) per thousand km of width.

Note that this is not affected by the amount of precession, since there is no precession when the ring is edge-on to the star.

6 Tension in the ring

The tension in the ring is $\tau = mr\omega^2/2\pi = mg/2\pi$. In other words, the material must be strong enough to suspend a column of itself of length r in a uniform gravitational field g. We have seen that r is 5 times the distance to the Moon, which in turn is about 10 times geosynchronous orbital radius.

Thus the demands on the material exceed by two orders of magnitude those necessary for the building of a space elevator on Earth.

There is also a tension in the ring due to the tidal force. This is:

$$\int_{\alpha=-\pi/2}^{\pi/2} \frac{m\,d\alpha}{2\pi} 2\Omega^2 r \cos\alpha = \frac{2mr\Omega^2}{\pi}$$

The ratio of tidal tension to spin tension is therefore $4\Omega^2/\omega^2 = 2/y^2$. When this ratio is a significant fraction of 1, that is, when the spin period is a significant fraction of half the orbital period, the ring will become significantly elongated along the orbital radius, assuming the material has negligible stiffness. How does the ring behave under these conditions?

7 Variation of precession with attitude

The precession varies with the attitude of the ring. When the ring is edge-on to the star, the tide exerts no torque. At 90 degrees to those configurations, the torque takes its maximum at the value computed above. In between, when the intersection line of the ring and the orbit plane make an angle γ with the orbit radius, the torque is given by an integral similar to the previous one, in which the subterm $r \sin\alpha \cos\theta$ is replaced by $r \sin\alpha \sin\gamma(\cos\theta \sin\alpha - \cos\alpha)$. Writing $T_{\max}$ for the value of T computed earlier, we obtain

$$T_\gamma = T_{\max} \sin\gamma \tag{18}$$

The torque vector is always parallel to the orbit. We adopt the convention that a positive value means a torque vector in the same direction as the orbital motion. This requires γ to be defined as increasing when the ring undergoes a translation in the direction of orbital motion, and zero at the edge-on moment when the leading edge of the ring is above the orbital plane. “Above” means the side to which the orbital rotation vector points.

What will be the effect of this varying torque on the plane of the ring and its spin velocity? Mathematically, the relationship is simply:

$$\dot{\mathbf{J}} = \mathbf{T}_\gamma \tag{19}$$

where $\dot{\mathbf{J}}$ and $\mathbf{T}_\gamma$ are the vector quantities. $\mathbf{T}_\gamma$ depends on both γ and θ, which both depend on the direction of J. Thus we can obtain a rather complicated differential equation for J.

In the limit of low precessional torque (defined as $T_{\max}/\Omega \ll J$, i.e. the maximum torque applied over one year would make only a small change to J), we can consider instead the average torque over a single orbit. This is $T_{\max}/2$, directed towards the star from the position of the ring at time 0. After one orbit, J will be changed by $T_{\max}/2\Omega$ in that direction. This rotates

J by an angle $T_{\max}/2\Omega J$ (to first order), which is equal to $\frac{\Omega}{2\omega}\sin\theta\cos\theta$. This in turn rotates the intersection of the ring plane with the orbital plane by an angle $\frac{\Omega}{2\omega}\cos\theta$ in the opposite direction to the orbital spin. For this precession of the eclipses to go once round the orbit therefore takes a number of orbits equal to $4\pi\omega/\Omega\cos\theta = \frac{4\pi}{\cos\theta}y$, or upwards of 12 times as many years as there are days in a year. For the earth-like Orbital, this is about 5,500 years.

The precise long-term evolution of the system remains to be studied. Can it be set up in such as way that the tilt and spin vary only within small bounds indefinitely, or does it tend to some limit, or can it tumble chaotically?

To be continued.